Laplace approximation for rough differential equation driven by fractional Brownian motion

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Abstract

We consider a rough differential equation indexed by a small parameter $\varepsilon > 0$. When the rough differential equation is driven by fractional Brownian motion with Hurst parameter H $(1/4 < H < 1/2)$, we prove the Laplace-type asymptotics for the solution as the parameter ε tends to zero.

1   Introduction

The rough path theory was invented by T. Lyons in [26] and summarized in a book [28] with Z. Qian. See also [24, 14, 27]. Roughly speaking, a rough path is a path coupled with its iterated integrals. T. Lyons generalized the line integral of one-form along a path to the one along a rough path. This is a pathwise integral theory and no probability measure is involved. In a natural way, an ordinary differential equation (ODE) is generalized. This is called a rough differential equation (RDE) in this paper. The corresponding Itô map is not only everywhere defined, but is also locally Lipschitz continuous with respect to the topology of geometric rough path space (Lyons' continuity theorem). If a Wiener-like measure is given on the geometric rough path space, or, in other words, if Brownian rough path is mapped by the Itô map, then the solution of corresponding stochastic differential equation (SDE) of Stratonovich-type is recovered via rough paths. In order to investigate the Brownian motion, one only needs the double integral (i.e., the second level path) as well as the path itself (i.e., the first level path). In short, we can obtain the solution of an SDE as the image of a continuous map. This is basically impossible in the framework of the usual stochastic calculus. Recall that, in the usual stochastic calculus, stochastic integrals and SDEs are defined by the martingale integration theory, which is quite probabilistic by definition. Therefore, those objects have no pointwise meaning.

Brownian motion and Brownian rough path are most important and were studied extensively. There may be other stochastic processes (i.e., probability measures on the usual path space), however, which can be lifted to probability measures on the geometric rough path space. The most typical example is the d-dimensional fractional Brownian motion (fBm) $(w_t^H)_{0 \le t \le 1} = (w_t^{H,1}, \ldots, w_t^{H,d})_{0 \le t \le 1}$ with Hurst parameter $H \in (1/4, 1/2]$ (see Coutin-Qian [9]). Recall that, when $H = 1/2$, it is the Brownian motion. It is worth noting that, if $H \in (1/4, 1/3]$, the third level path plays a role, unlike the Brownian motion case. The Schilder-type large deviation for the lift of scaled fBm was proved by Millet and Sanz-Sole [29]. Combined with Lyons' continuity theorem and the contraction principle, this fact implies that the solution of an RDE driven by the lift of scaled fBm also satisfies large deviation.

According to [8, 7], there are several types of path integrals along fBm, namely, (1) deterministic or pathwise integral (2) integral with generalized covariation, (3) the divergence operator in the sense of the Malliavin calculus, (4) White noise approach. Clearly, the rough path approach belongs to the first category.

More precisely, we consider the following RDE; for $\varepsilon > 0$,

$$dY_t^\varepsilon = \sigma(Y_t^\varepsilon) \varepsilon dW_t^H + \beta(\varepsilon, Y_t^\varepsilon) dt, \qquad Y_0^\varepsilon = 0. \qquad (1.1)$$

Here, W^H is the fractional Brownian rough path (fBrp) i.e., the lift of fBm w^H and $\sigma \in C_b^\infty(\mathbf{R}^n, \mathrm{Mat}(n, d))$ and $\beta \in C_b^\infty([0, 1] \times \mathbf{R}^n, \mathbf{R}^n)$. Note that C_b^∞ denotes the set of bounded smooth functions with bounded derivatives.

The main pupose of this paper is to prove the Laplace approximation for (the first level path of) Y^ε as $\varepsilon \searrow 0$. The precise statement is in Theorem 2.1 below. Apparently, in none of (1)–(4), has the Laplace approximation been proved for the solution of SDE (or RDE) driven by the scaled fBm. Note that it is a precise asymptotics of the large deviation. In this paper, we will prove it in the framework of the rough path theory for $H \in (1/4, 1/2)$.

The history of this kind of problem is long. A partial list could be as follows. First, Azencott [4] showed this kind of asymptotics for finite dimensional SDEs, which is followed by Ben Arous [6]. There are similar results for infinite dimensional SDEs (e.g., Albeverio-Röckle-Steblovskaya [3]) as well as SPDEs (e.g., Rovira-Tindel [33]). In the framework of the Malliavin calculus, there are deep results on the asymptotics of the generalized expectation of generalized Wiener functionals (Takanobu-Watanabe [34], Kusuoka-Stroock [21, 22], Kusuoka-Osajima [20]), which have applications to the asymptotics for the heat kernels on Riemannian manifolds.

In the framework of the rough path theory, Aida studied this problem for finite dimensional Brownian rough paths and gave a new proof for the results in [4, 6]. The same problem for infinite dimensional Brownian rough paths was studied in [16, 18], which has an application to Brownian motion over loop groups.

The organization of this paper is as follows: In Section 2, we give a precise statement of our main result. In Section 3, we review the rough path theory and fractional Brownian

rough path. In Section 4, we prove the Hilbert-Schimdt property of the Hessian of the Itô map restricted on the Cameron-Martin space of fBm. For those who understand the proof of Laplace approximation for Brownian rough path as in [2, 16, 18], this is the most difficult part, because the Cameron-Martin space of fBm is not understood very well. However, thanks to Friz-Victoir's result (Proposition 3.4), such Cameron-Martin paths are Young integrable and therefore the Hessian is computable. In Section 5 we give a probabilistic representation of (the stocastic extension of) the Hessian. In Section 6 we give a proof of the main theorem. In Section 7 we consider the Laplace approximation for an RDE, which involves a fractional order terms of $\varepsilon > 0$. This has an application to the short time asymptotics of integral quantities of the solution of a fixed RDE driven by fBm. (Similar problems were studied in [5, 31]). In Appendix (1) we explain the "shift" and the "pairing" on the geometric rough path space, which is actually a well-known fact. (2) we show that fractional Brownian rough path is scale invariant as fBm is. This fact does not seem completely obvious since the definition of the lift of fBm is related to the dyadic partitions. Although one can easily guess it, we still need a proof.

2   Assumption and Main Result

In this section we state our main results in this paper. Throughout this paper, the time interval is $[0, 1]$ except otherwise stated. Let $1/4 < H < 1/2$ and let $\mathcal{H}^H$ be the Cameron-Martin subspace of the d-dimensional fBm $(w_t^H)_{0 \le t \le 1}$. By Friz-Victoir's result, which will be explained in Proposition 3.4 below, $k \in \mathcal{H}^H$ is of finite q-variation for any $(H + 1/2)^{-1} < q < 2$. Hence, the following ODE makes sense in the q-variational setting in the sense of the Young integration;

$$dy_t = \sigma(y_t)dk_t + \beta(0, y_t)dt, \qquad y_0 = 0.$$

Note that y is again of finite q-variation and we will write $y = \Psi(k)$.

Now we set the following assumptions. In short, we assume that there is only one point that attains minimum of F_Λ and the Hessian at the point is non-degenerate. These are typical assumptions for Laplace's method of this kind. The space of continuous paths in $\mathbf{R}^n$ with finite p'-variation starting at 0 is denoted by $C_0^{p'-var}(\mathbf{R}^n)$. Note that the self-adjoint operator A in the fourth assumption turns out to be Hilbert-Schmidt in Theorem 4.1 below.

(H1): F and G are real-valued bounded continuous function on $C_0^{p'-var}(\mathbf{R}^n)$ for some $p' > 1/H$.

(H2): The function $F_\Lambda := F \circ \Psi + \| \cdot \|_{\mathcal{H}^H}^2 / 2$ attains its minimum at a unique point $\gamma \in \mathcal{H}^H$. We will write $\phi^0 = \Psi(\gamma)$.

(H3): F and G are $n + 3$ and $n + 1$ times Fréchet differentiable on a neighborhood $U(\phi^0)$ of $\phi^0 \in C_0^{p'-var}(\mathbf{R}^n)$, respectively. Moreover, there are positive constants $M_1, M_2, \dots$ such

that

$$
\begin{aligned}
|\nabla^j F(\eta)\langle z, \ldots, z\rangle| &\le M_j \|z\|_{p'-var}^j, &&(j = 1, \ldots, n+3) \\
|\nabla^j G(\eta)\langle z, \ldots, z\rangle| &\le M_j \|z\|_{p'-var}^j, &&(j = 1, \ldots, n+1)
\end{aligned}
$$

hold for any $\eta \in U(\phi^0)$ and $z \in C_0^{p'-var}(\mathbf{R}^n)$.

(H4): At the point $\gamma \in \mathcal{H}^H$, the bounded self-adjoint operator A on $\mathcal{H}^H$, which corresponds to the Hessian $\nabla^2(F \circ \Psi)(\gamma)|_{\mathcal{H}^H \times \mathcal{H}^H}$, is strictly larger than $-\mathrm{Id}_{\mathcal{H}^H}$ (in the form sense).

Under these assumptions, the following Laplace-type asymptotics holds. (Below, $Y^{\varepsilon,1} = (Y^\varepsilon)^1$ denotes the first level path of Y^ε);

Theorem 2.1 *Let the coefficients $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ and $\beta : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞. Then, under Assumtions **(H1)** – **(H4)**, we have the following asymptotic expansion as $\varepsilon \searrow 0$; there are real constants c and $\alpha_0, \alpha_1, \ldots$ such that*

$$
\begin{aligned}
&\mathbb{E}\big[G(Y^{\varepsilon,1}) \exp\big(-F(Y^{\varepsilon,1})/\varepsilon^2\big)\big] \\
&= \exp(-F_\Lambda(\gamma)/\varepsilon^2) \exp(-c/\varepsilon) \cdot \big(\alpha_0 + \alpha_1 \varepsilon + \cdots + \alpha_m \varepsilon^m + O(\varepsilon^{m+1})\big)
\end{aligned}
$$

for any $m \ge 0$.

Remark 2.2 *The only reason for the boundedness assumption for σ and b is for safety. It is an important and difficult problem whether Lyons' continuity theorem holds for unbounded coefficients under a mild growth condition. (One of such attempts can be found in [15]). If we have such an extension of the continuity theorem, then Theorem 2.1 could easily be generalized because localization around γ is crucially used in the proof (see Section 6 below).*

3 A review of fractional Brownian rough paths

In this section we recall that d-dimensional fBm $(w_t^H)_{0 \le t \le 1}$ with Hurst parameter $H \in (1/4, 1/2)$ can be lifted as a random variable on the geometric rough path space $G\Omega_p(\mathbf{R}^d)$ for $1/H < p < [1/H] + 1$ (Coutin-Qian [9] or Section 4.5 of Lyons-Qian [28]). When $H \in (1/4, 1/3]$, not only the first and the second level paths, but also the third level paths play a role.

3.1 Geometric rough paths, Lyons' continuity theorem and Taylor expansion of Itô maps

In this subsection, we recall definitions of geometric rough paths, and a rough differential equation (RDE) and Lyons' continuity theorem for Itô map. We also review (stochastic)

Taylor expansion for Itô maps around a "nice" path, which was shown in [17]. It plays a crucial role in the proof of the Laplace asymptotic expansion. Note that no probability measure is involved in this subsection. No new results are presented in this subsection.

Before inroducing the rough path space, let us first introduce some path spaces in the usual sense and norms on them. Let $\mathcal{V}$ be a real Banach space. Throughout this paper, we assume $\dim \mathcal{V} < \infty$ and the time interval is $[0,1]$. In almost all applications in later sections, either $\mathcal{V} = \mathbf{R}^d$ or $\mathcal{V} = \mathrm{Mat}(n,d)$ (the space of $n \times d$ matrices). Let

$$C = C([0,1], \mathcal{V}) = \{k : [0,1] \to \mathcal{V} \mid \text{continuous}\}$$

be the space of $\mathcal{V}$-valued continuous functions with the usual sup-norm. For $0 < \alpha < 1$, let $C^{\alpha - hldr}$ be the set of $k \in C$ such that

$$\|k\|_{\alpha - hldr}; = |k_0| + \sup_{0 \leq s < t \leq 1} \frac{|k_t - k_s|}{|t - s|^\alpha} < \infty.$$

Similarly, for $p \geq 1$, $C^{p - var}$ is the set of $k \in C$ such that

$$\|k\|_{p - var}; = |k_0| + \left(\sup_{\mathcal{P}} \sum_{i=1}^{n} |k_{t_i} - k_{t_{i-1}}|^p \right)^{1/p} < \infty,$$

where $\mathcal{P}$ runs over all the finite partition of $[0,1]$. If $p, q \geq 1$ with $1/p + 1/q > 1$ and $k \in C^{q - var}(L(\mathcal{V}, \mathcal{W}))$ and $l \in C^{p - var}(\mathcal{V})$ with $l_0 = 0$, then Young integral

$$\int_s^t k_u dl_u := \lim_{|\mathcal{P}| \searrow 0} \sum_{i=1}^{N} k_{t_{i-1}} (l_{t_i} - l_{t_{i-1}})$$

is well-defined. Here, $L(\mathcal{V}, \mathcal{W})$ is the set of linear maps from $\mathcal{V}$ to $\mathcal{W}$ and $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$ is a partition of $[s, t]$. Moreover, $t \mapsto \int_0^t k_u dl_u \in \mathbf{R}^n$ is of finite p-variation and $\| \int_0^{\cdot} k_u dl_u \|_{p - var} \leq const. \|k\|_{q - var} \|l\|_{p - var}$. More precisely, if there is a control function ω such that $|k_t - k_s| \leq \omega(s, t)^{1/q}$, $|l_t - l_s| \leq \omega(s, t)^{1/p}$, then

$$\left| \int_s^t k_u dl_u - k_s(l_t - l_s) \right| \leq const \cdot \omega(s, t)^{1/p + 1/q}.$$

In particular, if $\tilde{l} \in C^{p - var}(\mathcal{V})$ and $\tilde{k} \in C^{q - var}(\mathcal{W})$ with $1/p + 1/q > 1$, then $\int_s^t \tilde{k}_u \otimes d\tilde{l}_u$ is well-defined.

Next we introduce the Besov space $W^{\delta, p}$ for $p > 1$ and $0 < \delta < 1$. For a mesurable function $k : [0,1] \to \mathcal{V}$, set

$$\|k\|_{W^{\delta, p}} = \|k\|_{L^p} + \left(\iint_{[0,1]^2} \frac{|k_t - k_s|^p}{|t - s|^{1 + \delta p}} ds dt \right)^{1/p}. \tag{3.1}$$

The Besov space $W^{\delta, p}$ is the totality of k's such that $\|k\|_{W^{\delta, p}} < \infty$. When $1/p < \delta$, this Banach space is continuously imbedded in C and basically we only consider such a case.

The subspace of functions which start at 0 (i.e., $k_0 = 0$) is denoted by C_0, $C_0^{\alpha-hldr}$, etc. When we need to specify the range of functions, we write $C^{p-var}(\mathcal{V})$, $W_0^{\delta,p}(\mathcal{V})$, etc. (The domain is always $[0, 1]$ and, hence, is usually omitted.)

Now we introduce the geometric rough path space. Let $p \geq 1$ for a while. (In later sections, however, only the case $2 < p < 4$ will be considered.) Set $\triangle = \{(s, t) \mid 0 \leq s \leq t \leq 1\}$. The p-variation norm of a continuous map A form $\triangle$ to a real finite dimensional Banach space $\mathcal{V}$ is defined by

$$\|A\|_{p-var} = \left(\sup_{\mathcal{P}} \sum_{i=1}^{n} |A_{t_{i-1}, t_i}|_{\mathcal{V}}^p \right)^{1/p},$$

where $\mathcal{P}$ runs over all the finite partition of $[0, 1]$. A continuous map

$$X = (1, X^1, X^2, \ldots, X^{[p]}) : \triangle \to T^{[p]}(\mathcal{V}) = \mathbf{R} \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2} \cdots \oplus \mathcal{V}^{\otimes [p]}$$

is said to be a $\mathcal{V}$-valued rough path of roughness p if it satisfies the following conditions;

(a): For any $s \leq u \leq t$, $X_{s,t} = X_{s,u} \otimes X_{u,t}$, where $\otimes$ denotes the tensor operation in the truncated tensor algebra $T^{[p]}(\mathcal{V})$. In other words, $X_{s,t}^j = \sum_{i=0}^{j} X_{s,u}^i \otimes X_{u,t}^{j-i}$ for all $1 \leq j \leq [p]$. This is called Chen's identity.

(b): For all $1 \leq j \leq [p]$, $\|X^j\|_{p/j-var} < \infty$.

We usually omit the 0th component 1 and simply write $X = (X^1, \ldots, X^{[p]})$. The first level path of X is naturally regarded as an element in $C_0^{p-var}(\mathcal{V})$ by $t \mapsto X_{0,t}^1$. (We will abuse the notation to write $X^1 \in C_0^{p-var}(\mathcal{V})$, for example.) The set of all the $\mathcal{V}$-valued rough paths of roughness p is denoted by $\Omega_p(\mathcal{V})$. With the distance $d_p(X, Y) = \sum_{i=1}^{[p]} \|X^j - Y^j\|_{p/j-var}$, it becomes a complete metric space.

A $\mathcal{V}$-valued finite variational path $x \in C_0^{1-var}(\mathcal{V})$ is naturally lifted as an element of $\Omega_p(\mathcal{V})$ by the followiing iterated Stieltjes integral;

$$X_{s,t}^j = \int_{s \leq t_1 \leq \cdots \leq t_j \leq t} dx_{t_1} \otimes dx_{t_2} \otimes \cdots \otimes dx_{t_j}. \tag{3.2}$$

We say X is the smooth rough path lying above x. It is well-known that the injection $x \mapsto X \in \Omega_p(\mathcal{V})$ is continuous with respect to the 1-variation norm. The space of geometric rough path $G\Omega_p(\mathcal{V})$ is the closure of $C_0^{1-var}(\mathcal{V})$ with respect to d_p. Since $\mathcal{V}$ is separable, $G\Omega_p(\mathcal{V})$ is a complete separable metric space.

Let us recall some properties of q-variational path for $1 \leq q < 2$. For the facts presented below, see Section 3.3.2 in Lyons-Qian [28] or Inahama [17] for example. Since $k \in C_0^{q-var}(\mathcal{V})$ is Young integrable with respect to itself, the iterated integral in (3.2) still well-defined and k can be lifted to an element $K \in G\Omega_p(\mathcal{V})$ if $p \geq 2$. This injection $C_0^{q-var}(\mathcal{V}) \hookrightarrow G\Omega_p(\mathcal{V})$ is continuous.

For any $m = 1, 2, \ldots$ and any $k \in C_0(\mathcal{V})$, the mth dyadic piecewise linear approximation $k(m)$ is defined by

$$k(m)_t = k_{(l-1)/2^m} + 2^m (k_{l/2^m} - k_{(l-1)/2^m})(t - (l-1)/2^m), \qquad \text{if } t \in \left[\tfrac{l-1}{2^m}, \tfrac{l}{2^m} \right].$$

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If k is of q-variation ($q \geq 1$), then $k(m)$ converges to k in $(q+\varepsilon)$-variation norm for any $\varepsilon > 0$. It implies that, if $p \geq 2$ and $k \in C_0^{q-var}(\mathcal{V})$ for $1 \leq q < 2$, then $K(m)$ converges to K in $G\Omega_p(\mathcal{V})$.

Suppose that if $p \geq 2, 1 \leq q < 2$, and $1/p + 1/q > 1$, then the shift

$$(X, k) \in G\Omega_p(\mathcal{V}) \times C_0^{q-var}(\mathcal{V}) \mapsto X + K \in G\Omega_p(\mathcal{V})$$

is well-defined by Young integral and this map is continuous. Similarly,

$$(X, k) \in G\Omega_p(\mathcal{V}) \times C_0^{q-var}(\mathcal{W}) \mapsto (X, K) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$$

is well-defined and continuous. See subsection 8.1 in Appendix below.

Let $\mathcal{V}$ and $\mathcal{W}$ be two finite dimensional real Banach spaces and let $\sigma : \mathcal{W} \to L(\mathcal{V}, \mathcal{W})$ with some regularity condition, which will be specified later. We consider the following differential equation in the rough path sense (rough differential equation or RDE);

$$dY_t = \sigma(Y_t)dX_t, \qquad Y_0 = y_0 \in \mathcal{W}. \tag{3.3}$$

When there is a unique solution Y for given X, it is denoted by $Y = \Phi(X)$ and the map $\Phi : G\Omega_p(\mathcal{V}) \to G\Omega_p(\mathcal{W})$ is called the Itô map.

The following is called Lyons' continuity theorem (or universal limit theorem) and is most important in the rough path theory. (See Section 6.3, Lyons-Qian [28]. For a proof of continuity when the coefficient σ also varies, see Inahama [17] for example.)

Theorem 3.1 (i) *Let $p \geq 2$ and assume that $\sigma \in C_b^{[p]+1}(\mathcal{V}, \mathcal{W})$. Then, for given $X \in G\Omega_p(\mathcal{V})$ and a initial value $y_0 \in \mathcal{W}$, there is a unique solution $Y \in G\Omega_p(\mathcal{W})$ of RDE (3.3). Moreover, there is a constant $C_M > 0$ for $M > 0$ such that, if*

$$|y_0| \leq M, \quad \sum_{j=1}^{[p]} \|X^j\|_{p/j-var} \leq M, \quad \sum_{j=0}^{[p]+1} \sup_{y \in \mathcal{W}} \|\nabla^j \sigma(y)\| \leq M,$$

then, $\sum_{j=1}^{[p]} \|Y^j\|_{p/j-var} \leq C_M$.
(ii)   Keep the same assumption as above. Assume that $X_l \to X$ in $G\Omega_p(\mathcal{V})$ and $y_0^l \to y_0$ in $\mathcal{W}$ as $l \to \infty$. Assume further that $\sigma_l, \sigma \in C_b^{[p]+1}(\mathcal{V}, \mathcal{W})$ satisfy that

$$\sup_{l \geq 1} \sum_{j=0}^{[p]+1} \sup_{y \in \mathcal{W}} \|\nabla^j \sigma_l(y)\| \leq M$$

for some constant $M > 0$ and

$$\lim_{l \to \infty} \sum_{j=0}^{[p]+1} \sup_{|y|_{\mathcal{W}} \leq N} \|\nabla^j \sigma_l(y) - \nabla^j \sigma(y)\| = 0$$

for each fixed $N > 0$. Then, $Y_l \to Y$ in $G\Omega_p(\mathcal{W})$, where Y_l is the solution of RDE (3.3) corresponding to (X_l, y_0^l, σ_l).

In this paper, we consider the following RDE indexed by small parameter $\varepsilon > 0$. Let $\sigma \in C_b^\infty(\mathbf{R}^n, \mathrm{Mat}(n, d))$ and $\beta \in C_b^\infty([0, 1] \times \mathbf{R}^n, \mathbf{R}^n)$. For fixed $\varepsilon \in [0, 1]$, consider

$$dY_t^\varepsilon = \sigma(Y_t^\varepsilon)\varepsilon dX_t + \beta(\varepsilon, Y_t^\varepsilon)dt, \qquad Y_0^\varepsilon = 0. \tag{3.4}$$

(This is the same RDE as in (1.1)). If we define $\hat{\sigma}_\varepsilon : \mathbf{R}^n, \mathrm{Mat}(n, d + 1)$ by

$$\hat{\sigma}_\varepsilon(y)x' = \sigma(y)x + b(\varepsilon, y)x_{d+1}, \qquad x' = (x, x_{d+1}) \in \mathbf{R}^d \oplus \mathbf{R},$$

Then, $Y^\varepsilon = \hat{\Phi}_\varepsilon(\varepsilon X, \lambda)$. Here, $\lambda_t = t$ and $\hat{\Phi}_\varepsilon : G\Omega_p(\mathbf{R}^{d+1}) \to G\Omega_p(\mathbf{R}^n)$ is Itô map which corresponds to σ_ε. Note that σ_ε converges to $\sigma_{\varepsilon'}$ in the sense of Theorem 3.1 (ii) as $\varepsilon \to \varepsilon'$.

Now we consider the (stochastic) Taylor expansion around $\gamma \in C_0^{q-var}(\mathbf{R}^d)$ with $1/p + 1/q > 1$. Consider $\hat{\Phi}_\varepsilon(\varepsilon X + \gamma, \lambda)$, or equivalently the solution of the following RDE;

$$d\tilde{Y}_t^\varepsilon = \sigma(\tilde{Y}_t^\varepsilon)(\varepsilon dX_t + d\gamma_t) + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)dt, \qquad \tilde{Y}_0^\varepsilon = 0. \tag{3.5}$$

We will write $\phi^{(\varepsilon)} = (\tilde{Y}^\varepsilon)^1$ (the first level path). Note that $\hat{\Phi}_0(\gamma, \lambda)$ is lying above $\phi^0 = \Psi(\gamma) \in C_0^{q-var}(\mathbf{R}^n)$ which is defined by

$$d\phi_t^0 = \sigma(\phi_t^0)d\gamma_t + \beta(0, \phi_t^0)dt, \qquad \phi_0^0 = 0. \tag{3.6}$$

In the following theorem, we consider the asymptotic expansion of $\phi^{(\varepsilon)} - \phi^0$. By formally operating $(m!)^{-1}(d/d\varepsilon)^m|_{\varepsilon=0}$ the both sides of (3.5), we get an RDE for the mth term ϕ^m (see [17] for detail). Note that ϕ^m depends on X, γ (although γ is basically fixed in this paper), but independent of ε. (The superscript m does not denote the level of the path ϕ^m. Here we only consider the usual paths or the first level paths.)

In what follows, we will use the following notation; for a geometric rough path X of roughness p,

$$\xi(X) = \|X^1\|_{p-var} + \|X^2\|_{p/2-var}^{1/2} + \cdots + \|X^{[p]}\|_{p/[p]-var}^{1/[p]}. \tag{3.7}$$

Theorem 3.2 *Let $p \geq 2$, $1 \leq q < 2$ with $1/p + 1/q > 1$ and let the notations be as above. Then, for any $m = 1, 2, \ldots$, we have the following expansion;*

$$\phi^{(\varepsilon)} = \phi^0 + \varepsilon\phi^1 + \cdots + \varepsilon^m\phi^m + R_\varepsilon^{m+1}.$$

The maps $(X, \gamma) \in G\Omega_p(\mathbf{R}^d) \times C_0^{q-var}(\mathbf{R}^d) \mapsto \phi^k, R_\varepsilon^{m+1} \in C_0^{p-var}(\mathbf{R}^n)$ are continuous $(0 \leq k \leq m)$. Moreover, the following estimates **(a), (b)** *hold;*
(a) *For any $r_1 > 0$, there exists $C_1 > 0$ which depends only on r_1 such that, if $\|\gamma\|_{q-var} \leq r_1$, then $\|\phi^k\|_{p-var} \leq C_1(1 + \xi(X))^k$ holds.*
(b) *For any $r_2, r_3 > 0$, there exists $C_2 > 0$ which depends only on r_2, r_3 such that, if $\|\gamma\|_{q-var} \leq r_2$ and $\xi(\varepsilon X) \leq r_3$, then $\|R_\varepsilon^{m+1}\|_{p-var} \leq C_2(\varepsilon + \xi(\varepsilon X))^{m+1}$ holds.*

3.2 Fractional Brownian rough paths

First we introduce fractional Brownian motion (fBm for short) of Hurst parameter H. There are several books and surveys on fBm (see [7, 8, 30], for example). In this paper we only consider the case $1/4 < H < 1/2$. A real-valued continuous stochastic process $(w_t^H)_{t \geq 0}$ starting at 0 is said to a fBm of Hurst parameter H if it is a centered Gaussian process with

$$\mathbb{E}[w_t^H w_s^H] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \qquad (s, t \geq 0)$$

This process has stationary increments $\mathbb{E}[(w_t^H - w_s^H)^2] = |t-s|^{2H}$ $(s, t \geq 0)$, and self-similarity, i.e., for any $c > 0$, $(c^{-H} w_{ct}^H)_{t \geq 0}$ and $(w_t^H)_{t \geq 0}$ have the same law. Note that $(w_t^{1/2})_{t \geq 0}$ is the standard Brownian motion. For $d \geq 1$, a d-dimensional fBm is defined by $(w_t^{H,1}, \ldots, w_t^{H,d})_{t \geq 0}$, where $w_t^{H,i}$ $(i = 1, \ldots, d)$ are independent one-dimensional fBm's. Its law μ^H is a probability measure on $C_0(\mathbf{R}^d)$. (Actually, it is a measure on $C_0^{p-var}(\mathbf{R}^d)$ for $p > 1/H$, or on $C_0^{\alpha-hldr}(\mathbf{R}^d)$ for $\alpha < H$).

Let K^H be the Volterra kernel given by

$$K^H(t, s) = \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)} F\Big(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{s}{t}\Big) \mathbf{1}_{\{s < t\}}.$$

Here, F denotes the Gauss hypergeometric function. For a d-dimensional standard Brownian motion $(b_t)_{t \geq 0}$, the process

$$t \mapsto \int_0^t K^H(t, s) db_s$$

becomes a d-dimensional fBm with Hurst parameter H (see Decreusefond-Üstsunel [10]).

Let $H \in (1/4, 1/2)$. The existence of fractional Brownian rough path (fBRP for short) was shown by Coutin-Qian [9] as an almost sure limit of $W^H(m)$ as $m \to \infty$, where $W^H(m)$ is the smooth rough path lying above $w^H(m) \in C_0^{1-var}(\mathbf{R}^d)$. More precisely, they proved

$$\mathbb{E}\Big[\sum_{m=1}^\infty \|W^H(m+1)^j - W^H(m)^j\|_{p/j-var}\Big] < \infty \qquad (1 \leq j \leq [p]).$$

In particular, $W^H(m)$ converges to W^H in L^1-sense, too. When $1/3 < H < 1/2$, $[p] = 2$ and when $1/4 < H \leq 1/3$, $[p] = 3$.

Now we prove a theorem of Fernique-type for fBRP for later use. Essentially the same theorem is shown in [12], but with respect to a different norm. Therefore, we give a proof here for readers' convenience by using a useful estimate in Millet and Sanz-Sole [29]. (The case $H = 1/2$ is shown in [16], for example.)

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Proposition 3.3 *Let $1/4 < H < 1/2$ and W^H be a d-dimensional fBRP as above.*
(1). Then, there exists a positive constant c such that

$$\mathbb{E}[\exp(c\xi(W^H)^2)] = \int_{G\Omega_p(\mathbf{R}^d)} \exp(c\xi(X)^2)\mathbb{P}^H(dX) < \infty,$$

where ξ is given in (3.7) and $\mathbb{P}^H$ denotes the law of W^H.
(2). For any $r > 0$ and $1 \le j \le [p]$, $\lim_{m\to\infty} \mathbb{E}[\|W^H(m)^j - W^{H,j}\|^r_{p/j-var}] = 0$.

Proof. In this proof, $c_1, c_2, \ldots$ are positive constants which may change from line to line. For a rough path X of roughness p and $\gamma > p - 1$, set

$$D_{j,p}(X,Y) = \left(\sum_{n=1}^{\infty} n^\gamma \sum_{l=1}^{2^n} |X^j_{(l-1)/2^n, l/2^n} - Y^j_{(l-1)/2^n, l/2^n}|^{p/j}\right)^{j/p}, \qquad (1 \le j \le [p]).$$

When $Y = 0$, we write $D_{j,p}(X) = D_{j,p}(X,Y)$ for simplicity. From Section 4.1 in Lyons-Qian [28], the following estimates hold;

$$
\begin{aligned}
\|X^1 - Y^1\|^p_{p-var} &\le c_1 D_{1,p}(X,Y)^p \\
\|X^2 - Y^2\|^{p/2}_{p/2-var} &\le c_1\left[D_{2,p}(X,Y)^{p/2} + D_{1,p}(X,Y)^{p/2}(D_{1,p}(X)^p + D_{1,p}(Y)^p)^{1/2}\right] \\
\|X^3 - Y^3\|^{p/3}_{p/3-var} &\le c_1\left[D_{3,p}(X,Y)^{p/3} + D_{2,p}(X,Y)^{p/3}(D_{1,p}(X)^p + D_{1,p}(Y)^p)^{1/3}\right. \\
&\quad + D_{1,p}(X,Y)^{p/3}(D_{2,p}(X)^{p/2} + D_{2,p}(Y)^{p/2})^{2/3} \\
&\quad \left. + D_{1,p}(X,Y)^{p/3}(D_{1,p}(X)^p + D_{1,p}(Y)^p)^{2/3}\right] \qquad (3.8)
\end{aligned}
$$

Proposition 2 in [29] states that, there is a sequence $\{a_m\}$ of positive numbers converging to 0 such that, for any $r > p$,

$$\mathbb{E}[D_{j,p}(W^H(m), W^H)^r]^{1/r} \le a_m r^{j/2}$$

holds. For simplicity, set $F_m = D_{j,p}(W^H(m), W^H)^{2/j}$. Then, from the above inequality,

$$\mathbb{P}(N < F_m) \le N^{-N}\mathbb{E}[F_m^N] \le c_2^N a_m^N.$$

for $N = 4, 5, \ldots$. Therefore,

$$
\begin{aligned}
\mathbb{E}[e^{cF_m}] &\le \sum_{N=0}^{\infty} e^{c(N+1)}\mathbb{P}(N < F_m \le N+1) \\
&\le (e^c + \cdots + c^{4c}) + e^c \sum_{N=4}^{\infty} e^{cN}\mathbb{P}(N < F_m) \\
&\le (e^c + \cdots + c^{4c}) + e^c \sum_{N=4}^{\infty} \exp[N(c + \log c_2 - \log a_m)].
\end{aligned}
$$

For given $c > 0$, there exists m_0 such that $m \geq m_0$ implies $c + \log c_2 - \log a_m < 0$. Thus, we obtain

$$\sup_{m \geq m_0} \mathbb{E}[e^{cF_m}] \leq \sup_{m \geq m_0} \mathbb{E}[\exp(cD_{j,p}(W^H(m), W^H)^{2/j})] < \infty.$$

On the other hand, it is easy to see that, for each fixed m_0, there is a constant $c'(m_0) > 0$ such that $D_{j,p}(W^H(m_0))^{1/j} \leq c'(m_0)\|w^H\|_\infty$. Hence, the usual Fernique theorem for Guassian measures applies and $D_{j,p}(W^H(m_0))^{1/j}$ is square exponentially integrable. Using (3.8) and the triangle inequality for $D_{j,p}$, we prove (1). In a simlar way, we see that

$$\sup_{m \geq 1} \mathbb{E}[D_{j,p}(W^H(m))^r] < \infty, \qquad \sup_{m \geq 1} \mathbb{E}[\|\|W^H(m)^j\|\|_{p/j-var}^r] < \infty.$$

This implies (2). ∎

Let us introduce Cameron-Martin subspace $\mathcal{H}^H$ of fBm. (i.e., $k \in C_0(\mathbf{R}^d)$ is an element of $\mathcal{H}^H$ if and only if μ^H and $\mu^H(\cdot + k)$ are mutually absolutely continuous.) For $h \in L^2 = L^2([0,1], \mathbf{R}^d)$, set

$$k_t = \int_0^t K^H(t,s)h_s ds.$$

It is known that $k := Uh \in \mathcal{H}^H$ and the map $h \mapsto k = Uh$ is unitary from L^2 to $\mathcal{H}^H$.

When $H = 1/2$, it is easy to see $k \in \mathcal{H}^{1/2}$ is of finite 1-variation. But, when $H \in (1/4, 1, 2)$, does $k \in \mathcal{H}^H$ have a similar nice property in terms of variation norm? The following theorem answers this question. As a result, $\mathcal{H}^H$ is continuously (and compactly) embedded in $G\Omega_p(\mathbf{R}^d)$ for $p \geq 2$.

Proposition 3.4 *(Friz-Victior [13])*
(i) *Let $0 < \delta < 1$ and $p \geq 1$ such that $\alpha = \delta - 1/p > 0$ and set $q = 1/\delta$. Then, we have a continuous embedding*

$$W^{\delta,p} \subset C^{q-var}, \qquad W^{\delta,p} \subset C^{\alpha-hldr}.$$

More precisely, for $h \in W^{\delta,p}$,

$$\omega(s,t) = \|h\|_{W^{\delta,p};[s,t]}^q (t-s)^{\alpha q}, \qquad 0 \leq s \leq t \leq 1$$

becomes a control function in the sense of Lyons-Qian [28], p. 16, and h is controlled by a constant multiple of ω (i.e., $|h_t - h_s| \leq const \times \omega(s,t)^{1/q}$).
(ii) *Let the Hurst parameter $H \in (0, 1/2)$. If $1/2 < \delta < H + 1/2$, then $\mathcal{H}^H \Subset W_0^{\delta,2}$ (compact embedding). Therefore, for any $\alpha \in (0, H)$ and $q \in ((H+1/2)^{-1}, 2)$,*

$$\mathcal{H}^H \Subset C_0^{\alpha-hldr}, \qquad \mathcal{H}^H \Subset C_0^{q-var}.$$

We give a theorem of Cameron-Martin type for fBRP W^H. (For BRP, see [16] for example.) Let $1/4 < H < 1/2$ and $1/H < p < [1/H] + 1$. Then, fBRP W^H exists on $G\Omega_p(\mathbf{R}^d)$ and its law is a probability measure on $G\Omega_p(\mathbf{R}^d)$. By Proposition 3.4, there exists $1 \leq q < 2$ such that $\mathcal{H}^H \Subset C_0^{q-var} \subset G\Omega_p(\mathbf{R}^d)$ and $1/p + 1/q > 1$. Hence, the shift $X \mapsto X + K$ for $k \in \mathcal{H}^H$ is well-defined in $G\Omega_p(\mathbf{R}^d)$, where K is the lift of k as usual.

Proposition 3.5 *Let $\varepsilon > 0$ and let $\mathbb{P}_\varepsilon^H$ be the law of εW^H. Then, for any $k \in \mathcal{H}^H$, $\mathbb{P}_\varepsilon^H$ and $\mathbb{P}_\varepsilon^H(\,\cdot\, + K)$ are mutually absolutely continuous and, for any bounded Borel function f on $G\Omega_p(\mathbf{R}^d)$,*

$$\int_{G\Omega_p(\mathbf{R}^d)} f(X + K)\mathbb{P}_\varepsilon^H(dX) = \int_{G\Omega_p(\mathbf{R}^d)} f(X)\exp\Big(\frac{1}{\varepsilon^2}\langle k, X^1\rangle - \frac{1}{2\varepsilon^2}\|k\|_{\mathcal{H}^H}^2\Big)\mathbb{P}_\varepsilon^H(dX).$$

Here, $\langle k, X^1\rangle$ is the measurable linear functional associated with $k \in \mathcal{H}^H = (\mathcal{H}^H)^*$ for the fBm $t \mapsto X_{0,t}^1$ (i.e., the element of the first Wiener chaos of the fBm X^1 associated with k).

Proof. Since $W^H(m) \to W^H$ in $G\Omega_p(\mathbf{R}^d)$ and $k(m) \to k$ in q-variation norm as $m \to \infty$, respectively, $W^H + K = \lim_{m\to\infty}[W^H(m) + K(m)]$. On the other hand, $W^H(m) + K(m)$ is the lift of $w^H(m) + k(m) = (w^H + k)(m)$. Hence, the problem reduces to the usual Cameron-Martin theorem for fBm w^H. ∎

In the end of this subsection we give a Schilder-type large deviation principle for the law of εW^H as $\varepsilon \searrow 0$. This was shown by Millet and Sanz-Sole [29] (and by Friz-Victoir [13, 12]).

Proposition 3.6 *Let $\mathbb{P}_\varepsilon^H$ be the law of εW^H as above $(1/4 < H < 1/2)$. Then, as $\varepsilon \searrow 0$, $\{\mathbb{P}_\varepsilon^H\}_{\varepsilon>0}$ satisfies a large deviation principle with a good rate function I, which is given by*

$$I(X) = \begin{cases} \frac{1}{2}\|k\|_{\mathcal{H}^H}^2 & \text{(if X is lying above $k \in \mathcal{H}^H$)}, \\ \infty & \text{(otherwise)}. \end{cases}$$

4 Hilbert-Schmidt property of Hessian

In this section we consider the Itô map restricted on the Cameron-Martin space $\mathcal{H}^H$ of the fBm with Hurst parameter $H \in (1/4, 1/2)$ and prove that its Hessian is symmetric Hilbert-Schmidt bilinear form.

Throughout this section we set $\beta_0(y) = \beta(0, y)$ for simplicity. Consider the following RDE;

$$dY_t = \sigma(Y_t)dX_t + \beta_0(Y_t)dt, \qquad Y_0 = 0. \tag{4.1}$$

The Itô map $X \in G\Omega_p(\mathbf{R}^d) \mapsto \hat{\Phi}_0(X, \lambda) = Y \in G\Omega_p(\mathbf{R}^n)$ restricted on the Cameron-Martin space $\mathcal{H}^H$ of fBm is denoted by Ψ, i.e., $\Psi(k) = \hat{\Phi}_0(K, \lambda)$ for $k \in \mathcal{H}^H$. Here, K is a geometric rough path lying above k and $\lambda_t = t$. (Since k is of finite q-variation for some $q < 2$, as we will see below, this is well-defined. Regularity of $k \in \mathcal{H}^H$ in p-variational setting is studied by Friz-Victoir [13]. Fortunately, h is of finite q-variation for some $q < 2$ and, hence, the Young integral is possible.)

The aim of this section is to prove the following therem. Let F and p' be as in Assumption **(H1)**.

Theorem 4.1 $\nabla^2(F \circ \Psi)(\gamma)\langle \cdot, \cdot \rangle$ *is a symmetric Hilbert-Schmidt bilinear form on* $\mathcal{H}^H$ *for any* $\gamma \in \mathcal{H}^H$.

Note that

$$
\begin{aligned}
\nabla^2(F \circ \Psi)(\gamma)\langle f, k \rangle &= \nabla F(\Psi(\gamma))\langle \nabla^2 \Psi(\gamma)\langle f, k \rangle \rangle \\
&\quad + \nabla^2 F(\Psi(\gamma))\langle \nabla \Psi(\gamma)\langle f \rangle, \nabla \Psi(\gamma)\langle k \rangle \rangle.
\end{aligned}
$$

ODEs for $\nabla \Psi(\gamma)\langle k \rangle$ and $\nabla^2 \Psi(\gamma)\langle f, k \rangle$ will be given in (4.3)–(4.5) below.

Now we set conditions on parameters. First we have the Hurst parameter $H \in (1/4, 1/2)$. Then, we can choose p and $q = \delta^{-1}$ such that

$$
\frac{1}{p'} \vee \frac{1}{[1/H]+1} < \frac{1}{p} < H, \quad \frac{3}{4} < \frac{1}{q} < H + \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{q} > 1, \quad \frac{1}{q} - \frac{1}{p} > \frac{1}{2} \tag{4.2}
$$

For example, $1/p = H - 2\varepsilon$ and $1/q = H + 1/2 - \varepsilon$ for sufficiently small $\varepsilon > 0$ satisfy (4.2). Indeed,

$$
\frac{1}{p} + \frac{1}{q} = 1 + 2(H - \frac{1}{4}) - 3\varepsilon, \qquad \frac{1}{q} - \frac{1}{p} = \frac{1}{2} + \varepsilon.
$$

For this p and $q = \delta^{-1}$, the fBm with the Hurst parameter H can be lifted to $G\Omega_p(\mathbf{R}^d)$ and its Cameron-Martin space $\mathcal{H}^H$ satisfies Proposition 3.4 above. In particular, the Young integral of $k \in \mathcal{H}^H$ with respect to itself is possible since $q < 2$. The shift and the pairing of $X \in G\Omega_p(\mathbf{R}^d)$ by $k \in \mathcal{H}^H$ can be defined since $1/p + 1/q > 1$. In what follows we always assume (4.2).

The Banach space $W^{\delta,p}$ is defined by (3.1). In Adams [1], its original definition is given by a kind of real interpolation (precisely, the trace space of J. L. Lions. see paragraph 7.35, [1]) of $W^{1,p}$ and $W^{0,p} = L^p$. Those are equivalent Banach spaces (paragraph 7.48, [1]). On the other hand, $L^{\delta,p}$ is defined by the complex interpolation of (the complexification of) $W^{1,p}$ and $W^{0,p} = L^p$, i.e., $L^{\delta,p} = [W^{1,p}, L^p]_{1-\delta}$. If $p = 2$, $L^{\delta,2}$ and (the complexification of) $W^{\delta,2}$ are equivalent Hilbert spaces (not unitarily equivalent, see paragraph 7.59, [1]). As a result , $L^{\delta,2}_{real}$ and $W^{\delta,2}$ are equivalent real Hilbert spaces, where $L^{\delta,2}_{real}$ is the subspace of $\mathbf{R}^d$-valued functions in $L^{\delta,2}$.

Theorem 4.2 *The following functions of* $t \in [0,1]$ *form an orthonormal basis of* $L^{\delta,2}_{real}$ *and of* $L^{\delta,2} = L^{\delta,2}_{real} \otimes \mathbf{C}$.

$$
\{1 \cdot \mathbf{e}_i \mid 1 \leq i \leq d\} \cup \{\frac{\sqrt{2}}{(1+n^2)^{\delta/2}} \cos(n\pi t)\mathbf{e}_i \mid n \geq 1, 1 \leq i \leq d\}
$$

Here, $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ *is the canonical orthonormal basis of* $\mathbf{R}^d$.

Proof. It is sufficient to prove the case $d = 1$. Note that

$$W^{1,2} = \{f = c_0 + \sum_{n=1}^{\infty} c_n \sqrt{2} \cos(n\pi t) \mid c_n \in \mathbf{C}, \ \sum_{n=0}^{\infty} (1 + n^2)|c_n|^2 < \infty\}$$

and $\|f\|_{W^{1,2}}^2 = \sum_{n=0}^{\infty} (1 + n^2)|c_n|^2$. Similarly,

$$L^2 = \{f = c_0 + \sum_{n=1}^{\infty} c_n \sqrt{2} \cos(n\pi t) \mid c_n \in \mathbf{C}, \ \sum_{n=0}^{\infty} |c_n|^2 < \infty\}$$

and $\|f\|_{L^2}^2 = \sum_{n=0}^{\infty} |c_n|^2$. Therefore, $W^{1,2}$ and L^2 are unitarily isometric to $l_2^{(1)}$ and $l_2^{(0)} = l_2$, respectively, where

$$l_2^{(\delta)} = \{\mathbf{c} = (c_n)_{n=0,1,2,\dots} \in \mathbf{C}^{\infty} \mid \|\mathbf{c}\|_{l_2^{(\delta)}}^2 = \sum_{n=0}^{\infty} (1 + n^2)^{\delta} |c_n|^2\}, \qquad (\delta \in \mathbf{R}),$$

Thus, the problem is reduced to the complex interpolation of two Hilbert spaces of sequences. A simple calculation shows that $[l_2^{(1)}, l_2]_{1-\delta} = l_2^{(\delta)}$. This implies

$$L^{\delta,2} = \{f = c_0 + \sum_{n=1}^{\infty} c_n \sqrt{2} \cos(n\pi x) \mid c_n \in \mathbf{C}, \ \sum_{n=0}^{\infty} (1 + n^2)^{\delta} |c_n|^2 < \infty\}$$

with $\|f\|_{L^{\delta,2}}^2 = \sum_{n=0}^{\infty} (1 + n^2)^{\delta} |c_n|^2$, which ends the proof. ∎

We compute p-variation norm of cosine functions. The following lemma is taken from Nate Eldredge's unpublished manuscripts [11]. Before stating it, we introduce some definitions. Let x be a one-dimensional continuous path with $x_0 = 0$. We say that $s \in [0,1]$ is a forward maximum (or forward minimum) if $x_s = \max x|_{[s,1]}$ (or $x_s = \min x|_{[s,1]}$, respectively). Suppose x is piecewise monotone with local extrema $\{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$. (For simplicity, we assume $s_0, s_2, \dots$ are local minima and $s_1, s_3, \dots$ are local maxima. The reverse case is easily dealt with by just replacing x with $-x$.) If $s_2, s_4, \dots$ are not only local minima but also forward minima, and $s_1, s_3, \dots$ are not only local maxima but also forward maxima, then we say x is jog-free. (Note that x_0 is not required to be a forward extremum.)

Proposition 4.3 Let $p \geq 1$. (i) *If a one-dimensional continuous path x with $x_0 = 0$ is jog-free with extrema $\{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$, then*

$$\|x\|_{p-var} = \left(\sum_{i=1}^{n} |x_{s_i} - x_{s_{i-1}}|^p \right)^{1/p}$$

(ii) *In particular, p-variation norm of $c_n(t) = \cos(n\pi t) - 1$ is given by $\|c_n\|_{p-var} = 2n^{1/p}$.*

Proof. (ii) is immediate from (i). We show (i). For a contiunous path y and a partition $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$, we set $V_{p,\mathcal{P}}(y) = \left(\sum_{i=1}^{n} |y_{t_i} - y_{t_{i-1}}|^p\right)^{1/p}$. Then, $\|y\|_{p-var} = \sup_{\mathcal{P}} V_{p,\mathcal{P}}(y)$. First, note that, if y is monotone increasing (or decreasing) on $[t_{i-1}, t_{i+1}]$, then it is easy to see that $V_{p,\mathcal{P}\setminus\{t_i\}}(y) \geq V_{p,\mathcal{P}}(y)$. In other words, intermediate points in monotone intervals should not be included.

Let x be jog-free with extrema $\mathcal{Q} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ as in the statement of (i) and let $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$ be a partition which does not include all the s_j's. We will show below that there exists an s_j such that $V_{p,\mathcal{P}\cup\{s_j\}}(x) \geq V_{p,\mathcal{P}}(x)$.

Let s_j be the first exteremum not contained in $\mathcal{P}$. (For simplicity we assume it is local and forward maximum.) Let t_i be the last element of $\mathcal{P}$ less than s_j. Then, $s_{j-1} \leq t_i \leq s_j \leq t_{i+1}$. Since x is increasing on $[s_{j-1}, s_j]$ and x_{s_j} is forward maximum,

$$x_{s_j} - x_{t_i} \geq x_{t_{i+1}} - x_{t_i}, \qquad x_{s_j} - x_{t_{i+1}} \geq x_{t_i} - x_{t_{i+1}}.$$

which yields that $|x_{s_j} - x_{t_i}|^p + |x_{s_j} - x_{t_{i+1}}|^p \geq |x_{t_{i+1}} - x_{t_i}|^p$. Therefore, $V_{p,\mathcal{P}\cup\{s_j\}}(x) \geq V_{p,\mathcal{P}}(x)$.

For any $\varepsilon > 0$, there exists $\mathcal{P}$ such that $V_{p,\mathcal{P}}(x) \geq \|x\|_{p-var} - \varepsilon$. First by adding all the s_j's, then by removing all the intermediate points (i.e., t_i's which are not one of s_j's), we get $V_{p,\mathcal{Q}}(x) \geq \|x\|_{p-var} - \varepsilon$. Letting $\varepsilon \searrow 0$, we complete the proof of (i). ∎

Now we calculate the Hessian of Ψ, which is defined in (4.1). For $q < 2$, ODE like (4.1) is well-defined in q-variation sense, thanks to the Young integral. The continuity of Ψ is well-known. Smoothness of the Itô map in $q(< 2)$-variation setting is studied in Li-Lyons [25]. The explicit form of the derivatives are obtained in a similar way to the case of (stochastic) Taylor expansion.

Let $q \in [1, 2)$ for a while and fix $\gamma \in C_0^{q-var}$. Then $\phi^0 = \Psi(\gamma)$ is also of finite q-variation, which takes values in $\mathbf{R}^n$. Set

$$d\Omega_t = \nabla\sigma(\phi_t^0)\langle \cdot, d\gamma_t\rangle + \nabla\beta_0(\phi_t^0)\langle \cdot \rangle dt.$$

Then, Ω is an $\mathrm{End}(\mathbf{R}^n)$-valued path of finite q-variation. Next, consider the following $\mathrm{End}(\mathbf{R}^n)$-valued ODE in q-variation sense;

$$dM_t = d\Omega_t \cdot M_t, \qquad M_0 = \mathrm{Id}_n.$$

Its inverse satisfies a similar ODE;

$$dM_t^{-1} = -M_t^{-1} \cdot d\Omega_t, \qquad M_0^{-1} = \mathrm{Id}_n.$$

Although the coefficients of these ODEs are not bounded, thanks to their special forms, a unique solution exists and the Itô map in the q-variational setting is locally Lipschitz continuous. More precisely, if γ is controlled by a control function ω, then M and M^{-1} are controlled by a constant multiple of $\hat{\omega}$, where $\hat{\omega}(s, t) = \omega(s, t) + (t - s)$.

Set $\chi(k) = (\nabla\Psi)(\gamma)\langle k \rangle$ for simplicity, which a continuous path of finite q-variation, again. Then, it satisfies an $\mathbf{R}^n$-valued ODE;

$$d\chi_t - \nabla\sigma(\phi_t^0)\langle\chi_t, d\gamma_t\rangle - \nabla\beta_0(\phi_t^0)\langle\chi_t\rangle dt = \sigma(\phi_t^0)dk_t, \qquad \chi_0 = 0. \tag{4.3}$$

From this, we can obtain an explicit expression as follows;

$$\chi(k)_t = (\nabla\Psi)(\gamma)\langle k \rangle_t = M_t \int_0^t M_s^{-1}\sigma(\phi_s^0)dk_s. \tag{4.4}$$

In a similar way, $\psi_t = \nabla^2\Psi(\gamma)\langle k, k \rangle_t$ satisfies the following ODE;

$$d\psi_t - \nabla\sigma(\phi_t^0)\langle\psi_t, d\gamma_t\rangle - \nabla\beta_0(\phi_t^0)\langle\psi_t\rangle dt = \nabla\sigma(\phi_t^0)\langle\chi(k)_t, dk_t\rangle$$
$$+\frac{1}{2}\nabla^2\sigma(\phi_t^0)\langle\chi(k)_t, \chi(k)_t, d\gamma_t\rangle + \frac{1}{2}\nabla^2\beta_0(\phi_t^0)\langle\chi(k)_t, \chi(k)_t\rangle dt, \qquad \psi_0 = 0. \tag{4.5}$$

From this and by polarization, we see that

$$2\nabla^2\Psi(\gamma)\langle f, k \rangle_t$$
$$= M_t \int_0^t M_s^{-1}\{\nabla\sigma(\phi_s^0)\langle\chi(f)_s, dk_s\rangle + \nabla\sigma(\phi_s^0)\langle\chi(k)_s, df_s\rangle\}$$
$$+M_t \int_0^t M_s^{-1}\{\nabla^2\sigma(\phi_t^0)\langle\chi(f)_s, \chi(k)_s, d\gamma_s\rangle + \nabla^2\beta_0(\phi_s^0)\langle\chi(f)_s, \chi(k)_s\rangle ds\}$$
$$= : V_1(f, k)_t + V_2(f, k)_t. \tag{4.6}$$

It is obvious that

$$(f, k) \in C_0^{q-var}(\mathbf{R}^d) \times C_0^{q-var}(\mathbf{R}^d) \mapsto \nabla^2\Psi(\gamma)\langle f, k \rangle \in C_0^{q-var}(\mathbf{R}^n)$$

is a symmetric bounded bilinear functional.

Lemma 4.4 *Let $1/4 < H < 1/2$ and choose p and q as in (4.2) Then, for any bounded linear functional $\alpha \in C_0^{p-var}(\mathbf{R}^n)^*$, the symmetric bounded bilinear form $\alpha \circ V_2\langle \cdot, \cdot \rangle$ on the Cameron-Martin space $\mathcal{H}^H$ is of trace class. In particular, if $p' \geq p$, $\nabla F(\phi^0) \circ V_2$ is of trace class for a Fréchet differentiable function $F : C_0^{p'-var}(\mathbf{R}^n) \to \mathbf{R}$. Moreover, $\alpha \circ V_2$ extends to a bounded bilinear form on $C_0^{p-var}(\mathbf{R}^d)$. A similar fact holds for $\nabla^2 F(\phi^0)\langle\chi(\cdot), \chi(\cdot)\rangle$, too.*

Proof. Since $t \mapsto M_t$ and $t \mapsto M_t^{-1}\sigma(\phi_t^0)$ are of finite q-variation, the map $h \mapsto \chi(h)$ extends to a bounded linear map from $C_0^{p-var}(\mathbf{R}^d)$ to $C_0^{p-var}(\mathbf{R}^n)$, thanks to the Young integral. By using the Young integral again, we see that $(h, k) \mapsto V_2(h, k)$ extends to a bounded bilinear map from $C_0^{p-var}(\mathbf{R}^d) \times C_0^{p-var}(\mathbf{R}^d)$ to $C_0^{q-var}(\mathbf{R}^n) \subset C_0(\mathbf{R}^d)$.

On the other hand, μ^H (the law of the fBm with the Hurst parameter H) is supported in $C_0^{p-var}(\mathbf{R}^d)$. In other words, $(\mathcal{X}, \mathcal{H}^H, \mu^H)$ is an abstract Wiener space, where $\mathcal{X}$ is the

closure of $\mathcal{H}^H$ with respect to the p-variation norm. (The author does not know whether $\mathcal{X} = C_0^{p-var}(\mathbf{R}^d)$, nor whether $C_0^{p-var}(\mathbf{R}^d)$ is separable.)

Therefore, $\alpha \circ V_2$ is a bounded bilinear form on an abstract Wiener space. By Goodman's theorem (Theorem 4.6, Kuo [19]), its restriction on the Cameron-Matin space is of trace class. ∎

Now we compute V_1.

Lemma 4.5 *Let $1/4 < H < 1/2$ and choose p and q as in (4.2) Then, for any bounded linear functional $\alpha \in C_0^{p-var}(\mathbf{R}^n)^*$, the symmetric bounded bilinear form $\alpha \circ V_1 \langle \cdot , \cdot \rangle$ on the Cameron-Martin space $\mathcal{H}^H$ is Hilbert-Schimidt. In particular, if $p' \geq p$, $\nabla F(\phi^0) \circ V_1$ is Hilbert-Schimidt for a Fréchet differentiable function $F : C_0^{p'-var}(\mathbf{R}^n) \to \mathbf{R}$. Moreover, if α_l is weak* convergent to α as $l \to \infty$ in $C_0^{p-var}(\mathbf{R}^n)^*$, then $\alpha_l \circ V_1$ converges to $\alpha \circ V_1$ as $l \to \infty$ in the Hilbert-Schimidt norm.*

The rest of this section is devoted to proving this lemma. An integration by parts yields that
$$V_2 \langle f, k \rangle = R_1 \langle f, k \rangle + R_1 \langle k, f \rangle - (R_2 \langle f, k \rangle + R_2 \langle k, f \rangle)$$
where
$$
\begin{aligned}
R_1 \langle f, k \rangle_t &= M_t \int_0^t M_s^{-1} \nabla \sigma(\phi_s^0) \langle \sigma(\phi_s^0) f_s, dk_s \rangle, \\
R_2 \langle f, k \rangle_t &= M_t \int_0^t M_s^{-1} \nabla \sigma(\phi_s^0) \langle M_s \int_0^s d[M_u^{-1} \sigma(\phi_u^0)] f_u, dk_s \rangle.
\end{aligned}
$$

Lemma 4.6 *Let R_2 be as above and $\alpha \in C_0^{p-var}(\mathbf{R}^n)^*$. Then, as a bilinear form on $\mathcal{H}^H$, $\alpha \circ R_2$ is of trace class. Moreover, if α_l is weak* convergent to α as $l \to \infty$ in $C_0^{p-var}(\mathbf{R}^n)^*$, then $\alpha_l \circ R_2$ converges to $\alpha \circ R_2$ as $l \to \infty$ in the Hilbert-Schimidt norm.*

Proof. We use the Young integral. Since $u \mapsto M_u^{-1} \sigma(\phi_u^0)$ is of finite q-variation, we see that
$$\| \int_0^\cdot d[M_u^{-1} \sigma(\phi_u^0)] f_u \|_{q-var} \leq c_1 \| M_\cdot^{-1} \sigma(\phi_\cdot^0) \|_{q-var} \| f \|_{p-var} \leq c_2 \| f \|_{p-var}.$$
Similarly, since $s \mapsto M_s^{-1} \nabla \sigma(\phi_s^0)$, M_s are of finite q-variation,
$$\| M_\cdot \int_0^\cdot M_s^{-1} \nabla \sigma(\phi_s^0) \langle M_s \int_0^s d[M_u^{-1} \sigma(\phi_u^0)] f_u, dk_s \rangle \|_{p-var} \leq c_3 \| f \|_{p-var} \| k \|_{p-var}. \qquad (4.7)$$

Thus, $(f, k) \mapsto R_2 \langle h, k \rangle$ is a bounded bilinear map from $C_0^{p-var}(\mathbf{R}^d) \times C_0^{p-var}(\mathbf{R}^d)$ to $C_0^{p-var}(\mathbf{R}^n) \subset C_0(\mathbf{R}^n)$. In particular, $\alpha \circ R_2$ is a bounded bilinear form on $C_0^{p-var}(\mathbf{R}^n)$. Again by Goodman's theorem (Theorem 4.6, [19]), its restriction on the Cameron-Matin space is of trace class.

Now we prove the convergence. Note that (4.7) still holds even when f or k do not start at 0. Consider the following continuous inclusions (See Proposition 3.4. Below, all the function space are $\mathbf{R}^d$-valued);

$$\mathcal{H}^H \hookrightarrow W_0^{\delta,2} \cong L_{0,real}^{\delta,2} \hookrightarrow L_{real}^{\delta,2} \hookrightarrow C^{q-var} \hookrightarrow C^{p-var},$$

where $\delta = 1/q$ and $\cong$ denotes isomorphism (but not unitary) of Hilbert spaces. Let us first consider $R_2|_{L_{real}^{\delta,2} \times L_{real}^{\delta,2}}$. We will show that, for an ONB $\{f_k\}_{k=1,2,\dots}$ of $L_{real}^{\delta,2}$, it holds that $\sum_{k,j=1}^{\infty} \|R_2\langle f_k, f_j\rangle\|_{p-var}^2 < \infty$. As in Theorem 4.2, we set $f_{0,i}(t) = 1 \cdot \mathbf{e}_i$ and $f_{m,i}(t) = (1+m^2)^{-\delta/2}\sqrt{2}\cos(m\pi t)\mathbf{e}_i$ $(m = 1, 2, \dots)$. By Proposition 4.3,

$$\|f_{m,i}\|_{p-var} \leq (1+m^2)^{-\delta/2}\sqrt{2}(1+2m^{1/p}) \leq c\left(\frac{1}{1+m}\right)^{\frac{1}{q}-\frac{1}{p}}$$

for some constant $c > 0$. From this and (4.7),

$$
\begin{aligned}
\sum_{i,i'=1}^{d} \sum_{m,m'=0}^{\infty} \|R_2\langle f_{m,i}, f_{m',i'}\rangle\|_{p-var}^2 &\leq c\sum_{i,i'=1}^{d}\sum_{m,m'=0}^{\infty}\|f_{m,i}\|_{p-var}^2\|f_{m',i'}\|_{p-var}^2 \\
&\leq c\sum_{m=0}^{\infty}\left(\frac{1}{1+m}\right)^{2(\frac{1}{q}-\frac{1}{p})}\sum_{m'=0}^{\infty}\left(\frac{1}{1+m'}\right)^{2(\frac{1}{q}-\frac{1}{p})} < \infty,
\end{aligned}
$$

because $1/q - 1/p > 1/2$. Here, the constant $c > 0$ may change from line to line.

By the Banach-Steinhaus theorem, $\|\alpha_l - \alpha\|_{C^{p-var,*}} \leq c$ for some constant $c > 0$. Hence,

$$|(\alpha_l - \alpha) \circ R_2\langle f_{m,i}, f_{m',i'}\rangle|^2 \leq \|\alpha_l - \alpha\|_{C^*}^2\|R_2\langle f_{m,i}, f_{m',i'}\rangle\|_C^2 \leq c^2\|R_2\langle f_{m,i}, f_{m',i'}\rangle\|_{p-var}^2$$

By the dominated convergence theorem, $\|\alpha_k \circ R_2 - \alpha \circ R_2\|_{\mathrm{HS}-L_{real}^{\delta,2}} \to 0$ as $k \to \infty$. (The norm denotes the Hilbert-Schmidt norm.) This implies that

$$\|\alpha_l \circ R_2 - \alpha \circ R_2\|_{\mathrm{HS}-\mathcal{H}^H} \leq \|\iota\|_{\mathrm{op}}\|\iota^*\|_{\mathrm{op}}\|\alpha_l \circ R_2 - \alpha \circ R_2\|_{\mathrm{HS}-L_{real}^{\delta,2}} \to 0$$

as $l \to \infty$, where $\iota : \mathcal{H}^H \hookrightarrow L_{real}^{\delta,2}$ denotes the inclusion. $\blacksquare$

Lemma 4.7 *Let R_1 be as above and $\alpha \in C_0^{p-var}(\mathbf{R}^n)^*$. Then, as a bilinear form on $\mathcal{H}^H$, $\alpha \circ R_1$ is Hilbert-Schmidt. Moreover, if α_l is weak* convergent to α as $l \to \infty$ in $C_0^{p-var}(\mathbf{R}^n)^*$, then $\alpha_l \circ R_1$ converges to $\alpha \circ R_1$ as $l \to \infty$ in the Hilbert-Schimidt norm.*

Proof. The proof is similar to the one for Lemma 4.6. It is sufficient to show that

$$\sum_{i,i'=1}^{d}\sum_{m,m'=0}^{\infty}\|R_1\langle f_{m,i}, f_{m',i'}\rangle\|_{p-var}^2 < \infty. \tag{4.8}$$

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In this proof, $c > 0$ is a constant which may change from line to line.

It is easy to see that, if $m \neq m'$,

$$
\begin{aligned}
\sqrt{2}\cos(m\pi t)d[\sqrt{2}\cos(m'\pi t)] &= -2m'\pi\cos(m\pi t)\sin(m'\pi t)dt \\
&= -m'\pi\{\sin((m'+m)\pi t) + \sin((m'-m)\pi t)\}dt \\
&= m'd\Big[\frac{\cos((m'+m)\pi t)}{m'+m} + \frac{\cos((m'-m)\pi t)}{m'-m}\Big],
\end{aligned}
$$

and that, if $m = m'$, $\sqrt{2}\cos(m\pi t)d[\sqrt{2}\cos(m\pi t)] = d[\cos(2m\pi t)]/2$.

In the following, fix i, i'. First, we consider the case $m = m'$.

$$
\begin{aligned}
R_1\langle f_{m,i}, f_{m,i'}\rangle_t &= M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i, \mathbf{e}_{i'}\rangle\sqrt{2}\frac{\cos(m\pi s)}{(1+m^2)^{1/2q}}d\Big[\frac{\sqrt{2}\cos(m\pi s)}{(1+m^2)^{1/2q}}\Big] \\
&= \frac{1/2}{(1+m^2)^{1/q}}M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i, \mathbf{e}_{i'}\rangle\, d[\cos(2m\pi s)].
\end{aligned}
$$

By the Young integral and Proposition 4.3, we see that

$$
\begin{aligned}
\|R_1\langle f_{m,i}, f_{m,i'}\rangle\|_{p-var}^2 &\leq \frac{c}{(1+m^2)^{2/q}}\|\cos(2m\pi\,\cdot\,) - 1\|_{p-var}^2 \\
&\leq \frac{cm^{2/p}}{(1+m^2)^{2/q}} \leq \frac{c}{(1+m)^{4/q-2/p}}.
\end{aligned}
$$

Since $4/q - 2/p > 1$,

$$
\sum_{m=0}^\infty \|R_1\langle f_{m,i}, f_{m,i'}\rangle\|_{p-var}^2 < \infty. \tag{4.9}
$$

Next we consider the case $m \neq m'$.

$$
\begin{aligned}
&R_1\langle f_{m,i}, f_{m',i'}\rangle_t \\
&= M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i, \mathbf{e}_{i'}\rangle\sqrt{2}\frac{\cos(m\pi s)}{(1+m^2)^{1/2q}}d\Big[\frac{\sqrt{2}\cos(m'\pi s)}{(1+m'^2)^{1/2q}}\Big] \\
&= \frac{m'}{(1+m^2)^{1/2q}(1+m'^2)^{1/2q}(m'+m)} \\
&\qquad\qquad \times M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i, \mathbf{e}_{i'}\rangle d[\cos((m'+m)\pi s)] \\
&\quad +\frac{m'}{(1+m^2)^{1/2q}(1+m'^2)^{1/2q}(m'-m)} \\
&\qquad\qquad \times M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i, \mathbf{e}_{i'}\rangle d[\cos((m'-m)\pi s)] \\
&=: \hat{R}_1^{i,i'}(m, m')_t + \hat{R}_2^{i,i'}(m, m')_t.
\end{aligned}
$$

By using the estimate for the Young integral again, we see that

$$
\begin{aligned}
\|\hat{R}_1^{i,i'}\langle f_{m,i}, f_{m,i'}\rangle\|_{p-var}^2 \;\leq\; & \frac{cm'^2}{(1+m^2)^{1/q}(1+m'^2)^{1/q}|m'+m|^2}\|\cos((m'+m)\pi\,\cdot\,)-1\|_{p-var}^2 \\
\leq\; & \frac{cm'^2|m'+m|^{2/p}}{(1+m^2)^{1/q}(1+m'^2)^{1/q}|m'+m|^2} \\
\leq\; & \frac{c|(m'+m)-m|^{2(1-1/q)}}{(1+|m|)^{2/q}|m'+m|^{2(1-1/p)}} \\
\leq\; & \frac{c}{(1+|m|)^{2/q}(1+|m'+m|)^{2(1/q-1/p)}} \\
& +\frac{c}{(1+|m|)^{4(1/q-1/2)}(1+|m'+m|)^{2(1-1/p)}}.
\end{aligned}
$$

It is easy to see that $2/q > 1$ and $2(1-1/p) > 1$ hold. From (4.2), $2(1/q-1/p) > 1$ and $4(1/q-1/2) > 1$. (The condition $1/q > 3/4$ is used here.) Therefore,

$$
\begin{aligned}
\sum_{0\leq m,m'<\infty, m\neq m'} & \|\hat{R}_1^{i,i'}\langle f_{m,i}, f_{m,i'}\rangle\|_{p-var}^2 \\
\leq\; & c\sum_{m,m'\in\mathbf{Z}}\Big(\frac{1}{(1+|m|)^{2/q}(1+|m'+m|)^{2(1/q-1/p)}} \\
& \qquad\qquad\qquad +\frac{1}{(1+|m|)^{4(1/q-1/2)}(1+|m'+m|)^{2(1-1/p)}}\Big) \\
=\; & c\sum_{m\in\mathbf{Z}}\frac{1}{(1+|m|)^{2/q}}\Big(\sum_{m'\in\mathbf{Z}}\frac{1}{(1+|m'+m|)^{2(1/q-1/p)}}\Big) \\
& +c\sum_{m\in\mathbf{Z}}\frac{1}{(1+|m|)^{4(1/q-1/2)}}\Big(\sum_{m'\in\mathbf{Z}}\frac{1}{(1+|m'+m|)^{2(1-1/p)}}\Big) < \infty. \qquad (4.10)
\end{aligned}
$$

In the same way as above,

$$
\sum_{0\leq m,m'<\infty, m\neq m'} \|\hat{R}_2^{i,i'}\langle f_{m,i}, f_{m,i'}\rangle\|_{p-var}^2 < \infty. \qquad (4.11)
$$

From (4.9), (4.10), and (4.11), we have (4.8), which competes the proof. ∎

5 A probabilistic representation of Hessian

Throughout this section we assume (4.2). Let $C_0 = C_0([0,1], \mathbf{R}^d)$ be the continuous path space starting at the origin and let μ be the Wiener measure. We regard $L^2 = L^2([0,1], \mathbf{R}^d)$ as the tangent space of C_0. (For example, Nualart's book [32] is written in this style. However, in most of the literature on the Malliavin Calculus, the tangent space is the Cameron-Martin space $\{k \in C_0 \mid k'$ is in $L^2\}$.) We define $\langle h, b\rangle = \int_0^1 h_t db_t$ for $h \in L^2$,

where $(b_t)_{0\le t\le 1}$ is the canonical realization of the Brownian motion. For cylinder function $F(b) = f(\langle h_1, b\rangle, \ldots, \langle h_m, b\rangle)$, where $f : \mathbf{R}^m \to \mathbf{R}$ is a bounded smooth function with bounded derivatives, we set

$$D_h F(b) = \sum_{j=1}^m \partial_j f(\langle h_1, b\rangle, \ldots, \langle h_m, b\rangle)(h_j, h)_{L^2}, \qquad h \in L^2.$$

and

$$DF(b) = \sum_{j=1}^m \partial_j f(\langle h_1, b\rangle, \ldots, \langle h_m, b\rangle)h_j.$$

Note that DF is an $L^2 = L^2([0,1], \mathbf{R}^d)$-valued function.

Let $(b_t)_{0\le t\le 1}$ be as above and set

$$w_t^H = \int_0^t K^H(t, s)db_s.$$

Then, $(w_t^H)_{0\le t\le 1}$ is a d-dimensional Brownian motion. Here, K^H is the Volterra kernel (see [10, 7, 8, 30]). Similarly, for $h \in L^2 = L^2([0,1], \mathbf{R}^d)$, set

$$k_t = \int_0^t K^H(t, s)h_s ds.$$

It is known that $k := Uh \in \mathcal{H}^H$ and the map $h \mapsto k = Uh$ is unitary from L^2 to $\mathcal{H}^H$. For simplicity we also write $w^H = Ub$. By closability of the derivative D,

$$D_h w_t^H = \int_0^t K^H(t, s)h_s ds = k_t = (Uh)_t.$$

Let $\mathcal{C}_n(\mu)$ and $\mathcal{C}_n(\mu^H)$ ($n = 0, 1, 2, \ldots$) be the nth Wiener chaos of b and w^H, respectively. It is well-known that $\mathcal{C}_n(\mu)$ are mutually orthogonal and $L^2(\mu) = \oplus_{n=0}^\infty \mathcal{C}_n(\mu)$. Similar facts hold for $\mathcal{C}_n(\mu^H)$, too. The map U induces a unitary isometry $\mathcal{C}_n(\mu) \cong \mathcal{C}_n(\mu^H)$ for all n. The second Wiener chaos $\mathcal{C}_2(\mu)$ is unitarily isometric with the space of symmetric Hilbert-Schmidt operators (or symmetric Hilbert-Schmidt bilinear forms) $L^2([0,1], \mathbf{R}^d) \otimes_{sym} L^2([0,1], \mathbf{R}^d)$ in a natural way.

Lemma 5.1 *Let V_1 be as in (4.6) and consider $V_1(w^H(m), w^H(m))_t$, where $w^H(m)$ denotes the mth dyadic polygonal approximation of w^H. Then, for $h, \hat{h} \in L^2$,*

$$\frac{1}{2}D_h V_1(w^H(m), w^H(m))_t = V_1(k(m), w^H(m))_t,$$

$$\frac{1}{2}D_{\hat{h}}D_h V_1(w^H(m), w^H(m))_t = V_1(k(m), \hat{k}(m))_t.$$

Here, $k = Uh, \hat{k} = U\hat{h} \in \mathcal{H}^H$. Moreover, as $m \to \infty$, the right hand sides of the above equations converge to

$$V_1(k, w^H)_t \qquad and \qquad V_1(k, \hat{k})_t$$

almost surely and in $L^2(\mu)$. (Note that the above quantities are well-defined since w^H is of finite p-variation and $k, \hat{k}$ is of finite q-variation with $1/p + 1/q > 1$. Since, $k, \hat{k}$ are of finite $(q - \varepsilon)$-variation for sufficinetly small $\varepsilon > 0$, $k(m), \hat{k}(m)$ converge to $k, \hat{k}$ in q-variation norm, respectively.)

Proof. On $[(l - 1)/2^m, l/2^m]$, $dw^H(m)_t = 2^n(w^H_{l/2^m} - w^H_{(l-1)/2^m})dt$. Therefore,

$$D_h dw^H(m)_t = 2^n D_h(w^H_{l/2^m} - w^H_{(l-1)/2^m})dt = 2^n(k_{l/2^m} - k_{(l-1)/2^m})dt = dk(m)_t.$$

From this, we see that

$$D_h \chi(w^H(m))_t = M_t \int_0^t M_s^{-1} \sigma(\phi_s^0) D_h dw^H(m)_s = M_t \int_0^t M_s^{-1} \sigma(\phi_s^0) dk(m)_s = \chi(k(m))_t.$$

Since $\|k(m) - k\|_{q-var}$ as $m \to \infty$ and $k \mapsto \chi(k)$ is bounded linear from $C_0^{q-var}(\mathbf{R}^d)$ to $C_0^{q-var}(\mathbf{R}^d)$, $\|\chi(k(m)) - \chi(k)\|_{q-var}$ as $m \to \infty$. (It is possible to replace q with $q - \varepsilon$ for sufficiently small $\varepsilon > 0$ so that $k \in C_0^{(q-\varepsilon)-var}$ still holds.) In a similar way,

$$
\begin{aligned}
\frac{1}{2} D_h V_1(w^H(m), w^H(m))_t &= M_t \int_0^t M_s^{-1} \big\{ \nabla\sigma(\phi_s^0)\langle D_h\chi(w^H(m))_s, dw^H(m)_s \rangle \\
&\qquad\qquad + \nabla\sigma(\phi_s^0)\langle \chi(w^H(m))_s, D_h dw^H(m)_s \rangle \big\} \\
&= M_t \int_0^t M_s^{-1} \big\{ \nabla\sigma(\phi_s^0)\langle \chi(k(m))_s, dw^H(m)_s \rangle \\
&\qquad\qquad + \nabla\sigma(\phi_s^0)\langle \chi(w^H(m))_s, dk(m)_s \rangle \big\} \\
&= V_1(k(m), w^H(m))_t.
\end{aligned}
$$

Since $\|w^H(m) - w^H\|_{p-var} \to 0$ as $m \to \infty$ almost surely and in $L^r(\mu)$ for any $r > 0$ (see [29]), $(1/2)D_h V_2(w^H(m), w^H(m))_t \to V_1(k, w^H)_t$ almost surely and in L^2. Finally,

$$(1/2)D_{\hat{h}} D_h V_2(w^H(m), w^H(m))_t = V_1(k(m), \hat{k}(m))_t,$$

which is non-random and clearly converges to $V_1(k, \hat{k})_t$ as $m \to \infty$. ∎

Proposition 5.2 *Let V_1 be as in (4.6) and consider $V_1(w^H(m), w^H(m))_t^i$, which is regarded as a function of $b = (b_t)_{0 \le t \le 1}$. Here, i denotes the ith component ($1 \le i \le n$). Then, for each fixed t, $V_1(w^H(m), w^H(m))_t^i$ converges almost surely and in $L^2(\mu)$ as $m \to \infty$. More precisely,*

$$\lim_{m \to \infty} V_1(w^H(m), w^H(m))_t^i = \Theta_t^i + \Lambda_t^i.$$

Here, Θ_t^i is an element in $\mathcal{C}_2(\mu)$ which corresponds to the symmetric Hilbert-Schmidt bilinear form $V_1(U\bullet, U\bullet)_t^i$ and $t \mapsto \Lambda_t^i = \lim_{m \to \infty} E[V_1(w^H(m), w^H(m))_t^i]$ is of finite p-variation.

Proof. First note that $V_1(x,x)$ has a rough path representation. Recall the (stochastic) Taylor expansion of Itô map (4.1) around γ. Then, $V_1(x,x)$ is calculated in computation of the second Taylor term. There is a continuous map $V' : G\Omega_p(\mathbf{R}^d) \to G\Omega_p(\mathbf{R}^n)$ such that $V_1(x,x) = V'(X)^1$ for all $x \in C_0^{q-var}(\mathbf{R}^d)$. Here, the superscript means the first level path and $X \in G\Omega_p(\mathbf{R}^d)$ is the lift of x. Moreover, since the integral that defines V_1 or V' in (4.6) is of second order, V' has the following property; there exists a constant $c > 0$ such that, for all $X, Y \in G\Omega_p(\mathbf{R}^d)$,

$$\|V'(X)^1\|_{p-var} \leq c(1 + \xi(X)^2),$$

$$\|V'(X)^1 - V'(Y)^1\|_{p-var} \leq c(1 + \xi(X)^c) \sum_{j=1}^{[p]} \|X^j - Y^j\|_{p/j-var}.$$

Here, $\xi(X) = \sum_{j=1}^{[p]} \|X^j\|_{p/j-var}^{1/j}$. From this, almost convergence of $V_1(w^H(m), w^H(m)) = V'(w^H(m)))^1$ to $V'(W^H))^1$ is obvious.

It is shown in [9] that $E[\|W^H(m)^j - W^{H,j}\|_{p/j-var}] \to 0$ as $m \to \infty$. From Proposition 3.3, $\sup_m E[\|W^H(m)^j\|_{p/j-var}^r] < \infty$ for any $r > 0$ and $1 \leq j \leq [p]$. Then, we easily see from these and Hölder's inequality that

$$E[\|V'(W^H(m))^1 - V'(W^H)^1\|_{p-var}^2] \to 0 \qquad \text{as } m \to \infty.$$

This implies the L^2-convergence. Since $V'(W^H) = \lim_{m\to\infty} V_1(w^H(m), w^H(m))$ is a $C_0^{p-var}(\mathbf{R}^n)$-valued random variable, $\|E[V'(W^H)^1]\|_{p-var} \leq E[\|V'(W^H)^1\|_{p-var}] < \infty$, which shows that Λ is of finite p-variation.

By Lemma 5.1 and the closability of the derivative operator D in $L^2(\mu)$,

$$\frac{1}{2}D_h V'(W^H)_t^{1,i} = V_1(Uh, w^H)_t^i, \qquad \frac{1}{2}D_{\hat{h}}D_h V'(W^H)_t^{1,i} = V_1(Uh, U\hat{h})_t^i.$$

where the superscript i denotes the ith component of $\mathbf{R}^n$. These equality imply that $V'(W^H)_t^{1,i} - E[V'(W^H)_t^{1,i}]$ is in $\mathcal{C}_2$ which corresponds to $V_1(U\bullet, U\bullet)_t^i$. ∎

Lemma 5.3 *Let $p' > p$ and $F : C_0^{p'-var}(\mathbf{R}^n)$ be a Fréchet differentiable function. Let*

$$\Theta_t = \lim_{m\to\infty} V_1(w^H(m), w^H(m))_t - E[\lim_{m\to\infty} V_1(w^H(m), w^H(m))_t]$$

be as in Proposition 5.2. Then, $\nabla F(\phi^0)\langle\Theta\rangle \in \mathcal{C}_2(\mu^H)$ which corresponds to the symmetric Hilbert-Schmidt bilinear form $\nabla F(\phi^0) \circ V_1 = \nabla F(\phi^0)\langle V_1(\bullet, \bullet)\rangle$ on $\mathcal{H}^H$.

Proof. Denote by g_K the element of $\mathcal{C}_2(\mu^H)$ which corresponds to a symmetric Hilbert-Schmidt bilinear form (or equivalently, operator) K and set

$$M := \{\alpha \in C_0^{p-var}(\mathbf{R}^n)^* \mid \alpha\langle\Theta(w)\rangle = g_{\alpha\circ V_1}(w) \quad \text{a.a.w } (\mu^H) \}.$$

Obviously, M is a linear subspace. Moreover, from Lemma 4.5, M is closed under weak*-limit. By Lemma 5.2, the evaluation map ev_t^i ($t \in [0,1], 1 \le i \le n$) defined by $\mathrm{ev}_t^i \langle y \rangle = y_t^i$ is in M. Denote by $\pi_m : C_0^{p-var}(\mathbf{R}^n) \to C_0^{p-var}(\mathbf{R}^n)$ be the projection defined by $\pi(m)y = y(m)$, where $y(m)$ is the mth dyadic piecewise linear approximation of $y \in C_0^{p-var}(\mathbf{R}^n)$. Note that $\nabla F(\phi^0)\langle \pi(m)y \rangle$ can be written as a linear combination of $y_{k/2^m}^i$ ($1 \le k \le 2^m, 1 \le i \le n$). Hence, $\nabla F(\phi^0) \circ \pi(m) \in M$. Since $p' > p$, $y(m) \to y$ in p'-variation norm. This implies that $\nabla F(\phi^0) \circ \pi(m) \to \nabla F(\phi^0)$ in the weak*-topology. Hence, $\nabla F(\phi^0) \in M$. ∎

Let A_1 be a self-adjoint Hilbert-Schimidt operator on $\mathcal{H}^H$ which corresponds to

$$\nabla F(\phi^0)\langle V_1(\bullet, \bullet)\rangle.$$

Then, $A - A_1$ is a self-adjoint Hilbert-Schimidt operator on $\mathcal{H}^H$ which corresponds to

$$\nabla F(\phi^0)\langle V_2(\bullet, \bullet)\rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle \chi(\bullet), \chi(\bullet)\rangle.$$

Obviously, this bilinear form extends to a one on $C_0^{p-var}(\mathbf{R}^d)$ and, hence, is of trace class by Goodman's theorem. See (4.6) for the definition of V_1, V_2. Combining these all, we see that

$$k \in \mathcal{H}^H \mapsto \langle Ak, k \rangle_{\mathcal{H}^H} = \nabla F(\phi^0)\langle \psi(k, k)\rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle \chi(k), \chi(k)\rangle$$

extends to a continuous map on $G\Omega_p(\mathbf{R}^d)$ and we denote it by $\langle AX, X \rangle$ for $X \in G\Omega_p(\mathbf{R}^d)$.

Lemma 5.4 *Let $\alpha \ge 1$ be such that $\mathrm{Id}_{\mathcal{H}^H} + \alpha A$ is strictly positive in the form sense. Then,*

$$\mathbb{E}[\exp(-\alpha\langle AW^H, W^H\rangle)] = \int_{G\Omega_p(\mathbf{R}^d)} \exp(-\alpha\langle AX, X\rangle)\mathbb{P}^H(dX) < \infty.$$

In particular, $e^{-\langle A\bullet, \bullet\rangle}$ is in $L^r(G\Omega_p(\mathbf{R}^d), \mathbb{P}^H)$ for some $r > 1$.

Proof. As a functional of w^H, $\langle (A - A_1)W^H, W^H \rangle$ is a sum of $\mathrm{Tr}(A - A_1)$ and the second order Wiener chaos corresponding to $A - A_1$. From Proposition 5.2 and Lemma 5.3, $\langle AW^H, W^H \rangle$ is a sum of a constant $\mathrm{Tr}(A - A_1) + \nabla F(\phi^0)\langle \Lambda \rangle$ and the second order Wiener chaos corresponding to A (which is denoted by Θ_A below). By well-known calculation,

$$\mathbb{E}[e^{-\alpha\Theta_A}] = \det_2(\mathrm{Id}_{\mathcal{H}^H} + \alpha A)^{-1/2},$$

where $\det_2$ denotes the Carlman-Fredholm determinant. ∎

6 A proof of Laplace approximation

6.1 Large deviation for the law of Y^ε as $\varepsilon \searrow 0$

In this section we prove the main theorem (Theorem 2.1). Let Y^ε be a solution of RDE (3.4). The law of $(Y^\varepsilon)^1 = Y^{\varepsilon,1}$ is the probability measure on $C_0^{p-var}(\mathbf{R}^n)$ for any $p > 1/H$. Then, by Theorem 3.1 and Proposition 3.6 we can use the contraction principle to see that the law of $\{Y^{\varepsilon,1}\}_{\varepsilon>0}$ satisfies large deviation as $\varepsilon \searrow 0$. The good rate function is given as follows;

$$I(y) = \begin{cases} \frac{1}{2}\inf\{\|k\|_{\mathcal{H}^H}^2 \mid y = \hat{\Phi}_0(k,\lambda)^1\} & \text{(if } y = \hat{\Phi}_0(k,\lambda)^1 \text{ for some } k \in \mathcal{H}^H), \\ \infty & \text{(otherwise).} \end{cases}$$

Here, $\hat{\Phi}_\varepsilon$ is the Itô map correspondng to RDE (3.4) and $\lambda_t = t$. For a bounded continuous function F on $C_0^{p-var}(\mathbf{R}^n)$, it holds that

$$\lim_{\varepsilon \searrow 0} \log \mathbb{E}[\exp(-F(Y^{\varepsilon,1})/\varepsilon^2)] = -\inf\{F(y) + I(y) \mid y \in C_0^{p-var}(\mathbf{R}^n)\}.$$

Now, let us consider Laplace's method, i.e., the precise asymptotic behaviour of the following integral

$$\begin{aligned} \mathbb{E}[\exp(-F(Y^{\varepsilon,1})/\varepsilon^2)] &= \int_{G\Omega_p(\mathbf{R}^d)} \exp(-F(\hat{\Phi}_\varepsilon(\varepsilon X,\lambda)^1)/\varepsilon^2)\mathbb{P}^H(dX) \\ &= \int_{G\Omega_p(\mathbf{R}^d)} \exp(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2)\mathbb{P}_\varepsilon^H(dX) \end{aligned}$$

as $\varepsilon \searrow 0$ under assumptions **(H1)**– **(H4)**.

Let $\gamma \in \mathcal{H}^H \subset G\Omega_p(\mathbf{R}^d)$ be the unique element at which $F(\hat{\Phi}_0(\,\cdot\,,\lambda)) + \|\cdot\|_{\mathcal{H}^H}^2/2$ attains minimum $(F_\Lambda(\gamma) =: a)$ as in **(H2)**. By a well-known argument, for any neighborhood of $O \subset G\Omega_p(\mathbf{R}^d)$ of γ, there exist positive constants δ, C such that

$$\int_{O^c} \exp(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2)\mathbb{P}_\varepsilon^H(dX) \leq Ce^{-(a+\delta)/\varepsilon^2}, \qquad \varepsilon \in (0,1].$$

This decays very fast and does not contribute to the asmptotic expansion.

6.2 Computation of α_0

In this subsection we compute the first term α_0 in the asymptotic expansion when $G \equiv 1$ (constant) and show $\alpha_0 > 0$. To do so, we need the (stochastic) Taylor expansion (Theorem 3.2) up to order $m = 2$. Once this is done, expansion up to higher order terms can be obtained rather easily.

For $\rho > 0$, set $U_\rho = \{X \in G\Omega_p(\mathbf{R}^d) \mid \xi(X) < \rho\}$, where ξ is given in (3.7). Then, taking $O = \gamma + U_\rho$, we see from the theorem of Cameron-Martin type (Proposition 3.5) that

$$\int_{\gamma+U_\rho} \exp(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2)\mathbb{P}_\varepsilon^H(dX)$$

$$= \int_{U_\rho} \exp(-F(\hat{\Phi}_\varepsilon(X+\gamma,\lambda)^1)/\varepsilon^2)\exp(-\frac{1}{\varepsilon^2}\langle\gamma,X^1\rangle - \frac{1}{2\varepsilon^2}\|\gamma\|_{\mathcal{H}^H}^2)\mathbb{P}_\varepsilon^H(dX)$$

$$= \int_{\{\xi(\varepsilon X)<\rho\}} \exp\left(-\frac{F(\phi^{(\varepsilon)})}{\varepsilon^2} - \frac{1}{\varepsilon}\langle\gamma,X^1\rangle - \frac{1}{2\varepsilon^2}\|\gamma\|_{\mathcal{H}^H}^2\right)\mathbb{P}^H(dX). \tag{6.1}$$

As we will see, $\langle\gamma,\cdot\rangle$ extends to a continuous linear functional on $C_0^{p-var}(\mathbf{R}^d)$ and in particular everywhere defined.

For sufficiently small ρ (i.e., $\rho \le \rho_0$ for some ρ_0), $\phi^{(\varepsilon)}$ is in the neighborhood of ϕ^0 as in Assumption **(H3)**. So, from Taylor expansion for F,

$$\begin{aligned}
F(\phi^{(\varepsilon)}) &= F(\phi^0) + \nabla F(\phi^0)\langle\phi^{(\varepsilon)} - \phi^0\rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle\phi^{(\varepsilon)} - \phi^0, \phi^{(\varepsilon)} - \phi^0\rangle \\
&\quad + \frac{1}{6}\int_0^1 d\theta\nabla^3 F(\theta\phi^{(\varepsilon)} + (1-\theta)\phi^0)\langle\phi^{(\varepsilon)} - \phi^0, \phi^{(\varepsilon)} - \phi^0, \phi^{(\varepsilon)} - \phi^0\rangle \\
&= F(\phi^0) + \nabla F(\phi^0)\langle\varepsilon\phi^1 + \varepsilon^2\phi^2\rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle\varepsilon\phi^1, \varepsilon\phi^1\rangle + Q_\varepsilon^3.
\end{aligned}$$

Here, the remainder term Q_ε^3 satisfies the following estimates; there exists a positive constant $C = C(\rho_0)$ such that

$$|Q_\varepsilon^3| \le C(\varepsilon + \xi(\varepsilon X))^3 \qquad \text{on the set } \{\xi(\varepsilon X) < \rho_0\}. \tag{6.2}$$

Note that C is independent of the choice of ρ ($\rho \le \rho_0$).

Now we compute the shoulder of exp on the right hand side of (6.1). Terms of order -2 are computed as follows;

$$-\frac{1}{\varepsilon^2}(F(\phi^0) + \frac{1}{2}\|\gamma\|_{\mathcal{H}^H}^2) = -\frac{a}{\varepsilon^2}.$$

Since $k \in \mathcal{H}^H \mapsto F(\Phi_0(k,\lambda)) + \|k\|_{\mathcal{H}^H}^2/2$ takes its minimum at $k = \gamma$, we see that

$$\langle k,\gamma\rangle_{\mathcal{H}^H} + \nabla F(\phi^0)\langle\chi(k)\rangle = 0,$$

where $\chi(k)$ is given by (4.3) or (4.4). By (4.4) and the Young integral, $k \mapsto \nabla F(\phi^0)\langle\chi(k)\rangle$ extends to a continuous linear map from $C_0^{p-var}(\mathbf{R}^d)$ and so does $\langle\gamma,\cdot\rangle_{\mathcal{H}^H}$. Hence, the measurable linear functional (i.e., the first Wiener chaos) associated with γ is this continuous extension. An ODE for $\phi^1 = \phi^1(k)$ is as follows;

$$d\phi_t^1 - \nabla\sigma(\phi_t^0)\langle\phi_t^1, d\gamma_t\rangle - \nabla_y\beta(0,\phi_t^0)\langle\phi_t^1\rangle dt = \sigma(\phi_t^0)dk_t + \nabla_\varepsilon\beta(0,\phi_t^0)dt, \qquad \phi_0^1 = 0. \tag{6.3}$$

Note that both ϕ^1 and χ extends to a continuous map from $G\Omega_p(\mathbf{R}^d)$. The difference $\theta_t^1 := \phi_t^1(X) - \chi_t(X)$ is independent of X (i.e., non-random), of finite variation, and satisfies

$$d\theta_t^1 - \nabla\sigma(\phi_t^0)\langle\theta_t^1, d\gamma_t\rangle - \nabla_y\beta(0, \phi_t^0)\langle\theta_t^1\rangle dt = \nabla_\varepsilon\beta(0, \phi_t^0)dt, \qquad \theta_0^1 = 0. \qquad (6.4)$$

Hence, terms of order -1 are computed as follows;

$$-\frac{1}{\varepsilon}\left(\nabla F(\phi^0)\langle\phi^1\rangle + \langle\gamma, X^1\rangle\right) = -\frac{\nabla F(\phi^0)\langle\theta^1\rangle}{\varepsilon}$$

Now we compute terms of order 0. The second term ϕ^2 in the expansion in Theorem 3.2 satisfies the following ODE (see [17] for example);

$$d\phi_t^2 - \nabla\sigma(\phi_t^0)\langle\phi_t^2, d\gamma_t\rangle - \nabla_y\beta(0, \phi_t^0)\langle\phi_t^2\rangle dt$$
$$= \nabla\sigma(\phi_t^0)\langle\phi_t^1, dX\rangle + \frac{1}{2}\nabla^2\sigma(\phi_t^0)\langle\phi_t^1, \phi_t^1, d\gamma_t\rangle + \frac{1}{2}\nabla_y^2\beta(\phi_t^0)\langle\phi_t^1, \phi_t^1\rangle dt$$
$$+ \nabla_y\nabla_\varepsilon\beta(\phi_t^0)\langle\phi_t^1\rangle dt + \frac{1}{2}\nabla_\varepsilon^2\beta(0, \phi_t^0)dt, \qquad \phi_0^2 = 0. \qquad (6.5)$$

Let χ and ψ be as in (4.3) and (4.5), respectively. By the same argument for (stochastic) Taylor expansion (Theorem 3.2), those extend to continuous maps from $G\Omega_p(\mathbf{R}^d)$ and we write $\chi(X)$ and $\psi(X) = \psi(X, X)$. If we set $\theta^2(X) := \phi^2(X) - \psi(X)$, then θ^2 satisfies the following ODE;

$$d\theta_t^2 - \nabla\sigma(\phi_t^0)\langle\theta_t^2, d\gamma_t\rangle - \nabla_y\beta(0, \phi_t^0)\langle\theta_t^2\rangle dt$$
$$= \nabla\sigma(\phi_t^0)\langle\theta_t^1, dX\rangle + \frac{1}{2}\nabla^2\sigma(\phi_t^0)\langle\theta_t^1, \theta_t^1, d\gamma_t\rangle + \nabla^2\sigma(\phi_t^0)\langle\theta_t^1, \chi_t, d\gamma_t\rangle$$
$$+ \frac{1}{2}\nabla_y^2\beta(\phi_t^0)\langle\theta_t^1, \theta_t^1\rangle dt + \nabla_y^2\beta(\phi_t^0)\langle\theta_t^1, \chi_t\rangle dt$$
$$+ \nabla_y\nabla_\varepsilon\beta(\phi_t^0)\langle\theta_t^1 + \chi_t\rangle dt + \frac{1}{2}\nabla_\varepsilon^2\beta(0, \phi_t^0)dt, \qquad \theta_0^2 = 0. \qquad (6.6)$$

Therefore, θ^2 is of first order, that is, for some constant $C > 0$, $\|\theta^2(X)\|_{p-var} \leq C(1+\xi(X))$ holds for any $X \in G\Omega_p(\mathbf{R}^d)$. In particular, by the Fernique-type theorem (Proposition 3.3), (a constant multiple of) θ^2 is exponentially integrable.

Hence, terms of order 0 on the shoulder of exp on the right hand side of (6.1) is as follows;

$$\nabla F(\phi^0)\langle\phi^2\rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle = \left[\nabla F(\phi^0)\langle\psi\rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle\chi, \chi\rangle\right] + \nabla F(\phi^0)\langle\theta^2\rangle$$
$$+ \frac{1}{2}\nabla^2 F(\phi^0)\langle\theta^1, \theta^1\rangle + \nabla^2 F(\phi^0)\langle\theta^1, \chi\rangle \qquad (6.7)$$

Note that the last three terms on the right hand side are dominated by $C(1 + \xi(X))$ and that the first term is $\psi(X, X) = \langle AX, X\rangle$ as in Lemma 5.4. By Proposition 3.3 and Lemma 5.4,

$$\exp\left(-\nabla F(\phi^0)\langle\phi^2\rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle\right) \in L^r(G\Omega_p(\mathbf{R}^d), \mathbb{P}^H) \qquad \text{for some } r > 1.$$

We easily see that, if $\varepsilon \le \rho$,

$$\mathbf{1}_{\{\xi(\varepsilon X) < \rho\}} \exp\left(-\nabla F(\phi^0)\langle \phi^2 \rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle \phi^1, \phi^1 \rangle\right) \exp(-\varepsilon^{-2} Q_\varepsilon^3)$$

$$\le \exp\left(-\nabla F(\phi^0)\langle \phi^2 \rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle \phi^1, \phi^1 \rangle\right) \exp[2\rho(1 + \xi(X))^2] \tag{6.8}$$

Note that if $\rho > 0$ is chosen sufficiently small, then the right hand side is integrable and independent of ε. (We determine ρ, here.) So, we may use the dominated convergence theorem to obtain that

$$\lim_{\varepsilon \searrow 0} \int_{\{\xi(\varepsilon X) < \rho\}} \exp\left(-\nabla F(\phi^0)\langle \phi^2 \rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle \phi^1, \phi^1 \rangle - \frac{1}{\varepsilon^2}Q_\varepsilon^3\right) \mathbb{P}^H(dX)$$

$$= \int_{G\Omega_p(\mathbf{R}^d)} \exp\left(-\nabla F(\phi^0)\langle \phi^2 \rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle \phi^1, \phi^1 \rangle\right) \mathbb{P}^H(dX).$$

By Lemma 5.4, the right hand side exists. Thus, we have computed (the asymptoitcs of) (6.1) up to α_0.

6.3 Asymptotic expansion up to any order

In this subsection we obtain the Laplace asymptotic expansion up to any order. Since this is a routine once α_0 is obtained, we only give a sketch of proof.

By combining the (stochastic) Taylor expansions for F, G, and $\phi^{(\varepsilon)}$, we get

$$F(\phi^{(\varepsilon)}) - F(\phi^0) \sim \varepsilon \eta^1 + \cdots + \varepsilon^n \eta^n + Q_\varepsilon^{n+1}, \qquad \text{as } \varepsilon \searrow 0$$
$$G(\phi^{(\varepsilon)}) - G(\phi^0) \sim \varepsilon \hat{\eta}^1 + \cdots + \varepsilon^n \hat{\eta}^n + \hat{Q}_\varepsilon^{n+1}, \qquad \text{as } \varepsilon \searrow 0$$

Here, the remainder terms $Q_\varepsilon^{n+1}, \hat{Q}_\varepsilon^{n+1}$ satisfy a simlar estimates to (6.2).

From this we see that

$$\int_{\gamma + U_\rho} G(\hat{\Phi}_\varepsilon(X, \lambda)^1) \exp(-F(\hat{\Phi}_\varepsilon(X, \lambda)^1)/\varepsilon^2) \mathbb{P}_\varepsilon^H(dX)$$

$$= e^{-a/\varepsilon^2} e^{-\nabla F(\phi^0)\langle \theta^1 \rangle/\varepsilon} \int_{\{\xi(\varepsilon X) < \rho\}} G(\phi^{(\varepsilon)})$$

$$\times \exp\left(-\nabla F(\phi^0)\langle \phi^2 \rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle \phi^1, \phi^1 \rangle\right) \exp(-Q_\varepsilon^3/\varepsilon^2) \mathbb{P}^H(dX) \tag{6.9}$$

can easily be expanded. Note that $Q_\varepsilon^3 = \varepsilon^3 \eta^3 + \cdots + \varepsilon^n \eta^n + Q_\varepsilon^{n+1}$. Thus we have shown the main theorem (Theorem 2.1).

7 Fractional order case: with an application to short time expansion

In this section we consider an RDE, which involves fractional order term of ε. As a result, fractional order term of ε appear in the asymptotic expansion. By time change, this has an application to the short time problems for the solutions of the RDE driven by fBrp.

Let $H \in (1/4, 1/3) \cup (1/3, 1/2)$. For simplicity, we consider the following RDE;

$$dY_t^\varepsilon = \sigma(Y_t^\varepsilon)\varepsilon dX_t + \varepsilon^{1/H}\hat{\beta}(Y_t^\varepsilon)dt, \qquad Y_0^\varepsilon = 0. \tag{7.1}$$

Here, σ is as in Theorem 2.1, but we assume that a C_b^∞-function $\hat{\beta} : \mathbf{R}^n \to \mathbf{R}^n$ and the drift term is of this special form in this case. Set $\beta(\varepsilon, y) = \varepsilon^{1/H}\hat{\beta}(y)$. We also consider the following RDE, which is independent of ε;

$$dV_t = \sigma(V_t)dX_t + \hat{\beta}(V_t)dt, \qquad V_0 = 0. \tag{7.2}$$

Basically, when we introduce randomness, we always set $X = W^H$ in (7.1) and (7.2). Then, by the scale invariance of W^H (see Proposition 8.1 below), $(V_{\varepsilon^{1/H}s, \varepsilon^{1/H}t})_{0 \le s \le t \le 1}$ and $(Y_{s,t}^\varepsilon)_{0 \le s \le t \le 1}$ have the same law. In particular, for each fixed $T \in (0, 1]$, the $\mathbf{R}^n$-valued random variables $V_{0,T}^1$ and $(Y^{T^H})_{0,1}^1$ have the same law. Therefore, the short time asymptotics for $V_{0,t}^1$ is related to the small asymptotics of $(Y^\varepsilon)^1$.

Let us fix some notations for fractional order expansions. For

$$M = \{n_1 + \frac{n_2}{H} \mid n_1, n_2 = 0, 1, 2, \ldots\},$$

let $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ be all elements of M in increasing order. More concretely, leading terms are as follows;

$$(\kappa_0, \kappa_1, \kappa_2, \ldots) = (0, 1, 2, \frac{1}{H}, 3, 1 + \frac{1}{H}, 4, 2 + \frac{1}{H}, 5 \wedge \frac{2}{H}, \ldots), \quad \text{if } H \in (1/3, 1/2),$$

$$(\kappa_0, \kappa_1, \kappa_2, \ldots) = (0, 1, 2, 3, \frac{1}{H}, 4, 1 + \frac{1}{H}, 5, \ldots,), \quad \text{if } H \in (1/4, 1/3). \tag{7.3}$$

As in the previous sections we write $Y^\varepsilon = \hat{\Phi}_\varepsilon(\varepsilon X)$, $\tilde{Y}^\varepsilon = \hat{\Phi}_\varepsilon(\varepsilon X + \gamma)$, and $\phi^\varepsilon = (\tilde{Y}^\varepsilon)^1$ for the solution of (7.1). By slightly modifying Theorem 3.2, we can prove the (stochastic) taylor expansion (around γ) for

$$\phi^{(\varepsilon)} = \phi^0 + \varepsilon^{\kappa_1}\phi^{\kappa_1} + \varepsilon^{\kappa_2}\phi^{\kappa_2} + \cdots + \varepsilon^{\kappa_m}\phi^{\kappa_m} + R_\varepsilon^{\kappa_{m+1}}.$$

In this case, ϕ^0 satisfies the following ODE (in q-variation sense);

$$d\phi_t^0 = \sigma(\phi_t^0)d\gamma_t, \qquad \phi_0^0 = 0. \tag{7.4}$$

Remark 7.1 *Although $(d/d\varepsilon)^m|_{\varepsilon=0}$ does not operate on the right hand side of the following (formal) ODE*

$$d\phi_t^{(\varepsilon)} = \sigma(\phi_t^{(\varepsilon)})d(\varepsilon X_t + \gamma) + \varepsilon^{1/H}\hat{\beta}(\phi_t^{(\varepsilon)})dt, \qquad \tilde{Y}_0^\varepsilon = 0, \tag{7.5}$$

the proof of expansion in [17], which is similar to Azencott's argument in [4], does not use ε-derivative and can be easily modified to our case.

Roughly and formally speaking, the proof goes as follows. First, combine

$$\phi^{(\varepsilon)} - \phi^0 = \varepsilon^{\kappa_1}\phi^{\kappa_1} + \cdots + \varepsilon^{\kappa_m}\phi^{\kappa_m} + \cdots$$

and the Taylor expansion of σ and $\hat{\beta}$ around ϕ_t^0. Next, pick up the terms of order α_m ($m = 1, 2, \ldots$). Then, we obtain a very simple ODE of first order for ϕ^{κ_m} recursively. This, in turn, can be used to rigorously define ϕ^{κ_m}. In the end, we prove growth of the remainder term is of expected order. (This part is non-trivial and requires much computation.) Note that this method can be used both in integer order and in fractional order cases.

In the same way as in the previous sections, we have the following modification of the main theorem (Theorem 2.1).

Theorem 7.2 *Let the coefficients $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n,d)$ and $\hat{\beta} : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞ and consider the RDE (7.1) with $X = W^H$, where $H \in (1/4, 1/3) \cup (1/3, 1/2)$. Then, under Assumptions **(H1)** – **(H4)**, we have the following asymptotic expansion as $\varepsilon \searrow 0$; there are real constants c and $\alpha_{\kappa_0}(= \alpha_0), \alpha_{\kappa_1}, \alpha_{\kappa_2}, \ldots$ such that*

$$\begin{aligned}
&\mathbb{E}\big[G(Y^{\varepsilon,1})\exp\big(-F(Y^{\varepsilon,1})/\varepsilon^2\big)\big] \\
&\quad = \exp(-F_\Lambda(\gamma)/\varepsilon^2)\exp(-c/\varepsilon)\cdot\big(\alpha_{\kappa_0} + \alpha_{\kappa_1}\varepsilon^{\kappa_1} + \cdots + \alpha_{\kappa_m}\varepsilon^{\kappa_m} + O(\varepsilon^{\kappa_{m+1}})\big)
\end{aligned}$$

for any $m \geq 0$.

Remark 7.3 *It is important to note that, in (7.3), indices up to degree two (i.e., $\kappa_0, \kappa_1, \kappa_2$) are the same as in the previous sections. The most difficult part of the proof of Theorem 2.1 is obtaining α_0 (or checking that $\alpha_0 \in (0, \infty)$ when $G \equiv 1$), in which the (stochastic) Taylor expansion of $\phi^{(\varepsilon)}$ up to ϕ^2 is used (see Subsection 6.2). Therefore, the proof in Subsection 6.2 holds true without modification in this case, too. Higher order terms are different in the fractional order case. But, the argument in Subsection 6.3 is simple anyway and can easily be modified. Thus, we can prove Theorem 7.2 without much difficulty.*

As a corollary, we have the following short time expansion. In the following, ev_1 denotes the evaluation map at time 1, i.e., $\mathrm{ev}_1(x) = x_1$ for an $\mathbf{R}^n$-valued path x.

Corollary 7.4 *Let the coefficients $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n,d)$ and $\hat{\beta} : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞ and consider the RDE (7.2) above with $X = W^H$, where $H \in (1/4, 1/3) \cup (1/3, 1/2)$. Let f and g be real-valued C_b^∞-functions on $\mathbf{R}^n$ such that $F := f \circ \mathrm{ev}_1$ and $G := g \circ \mathrm{ev}_1$ satisfy Assumptions **(H1)** – **(H4)**. Then, we have the following asymptotic expansion as $t \searrow 0$; there are real constants c and $\hat{\alpha}_{\kappa_0}(= \hat{\alpha}_0), \hat{\alpha}_{\kappa_1}, \hat{\alpha}_{\kappa_2}, \ldots$ such that*

$$\begin{aligned}
&\mathbb{E}\big[g(V_{0,t}^1)\exp\big(-f(V_{0,t}^1)/t^{2H}\big)\big] \\
&\quad = \exp(-F_\Lambda(\gamma)/t^{2H})\exp(-c/t^H)\cdot\big(\hat{\alpha}_{\kappa_0} + \hat{\alpha}_{\kappa_1}t^{\kappa_1 H} + \cdots + \hat{\alpha}_{\kappa_m}t^{\kappa_m H} + O(t^{\kappa_{m+1}H})\big)
\end{aligned}$$

for any $m \geq 0$.

Remark 7.5 *Very roughly speaking, in [5, 31], they studied the sort time asymptotics of the following quantity under mild assumptions;*

$$\mathbb{E}[g(V_{0,t}^1)].$$

If f is identically zero in Corollary 7.4, then it is somewhat similar to the short time problems studied in [5, 31]. (It does not seem to the author that either [5, 31] or the Corollary 7.4 implies the other.)

8 Appendix

8.1 The "shift" and "pairing" of geometric rough paths

In this subsection we recall the shift and pairing of geometric rough paths. Let $2 \le p < 4$ and $1 \le q < 2$ with $1/p + 1/q > 1$ and let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional Banach space (all the results in this subsection hold in infinite dimensional settings, too, if norms on the tensor spaces are appropriately chosen).

For $x, k \in C_0^{1-var}(\mathcal{V})$, the shift of $X \in G\Omega_p(\mathcal{V})$ (i.e., the lift of x) and k is naturally given by $X + K \in G\Omega_p(\mathcal{V})$ (i.e., the lift of $x + k \in C_0^{1-var}(\mathcal{V})$). Thanks to the Young integral, this extends to a continuous map

$$G\Omega_p(\mathcal{V}) \times C_0^{q-var}(\mathcal{V}) \ni (X, k) \mapsto X + K \in G\Omega_p(\mathcal{V}).$$

Similarly, for $x \in C_0^{1-var}(\mathcal{V})$ and $k \in C_0^{1-var}(\mathcal{W})$, the pairing $X \in G\Omega_p(\mathcal{V})$ and k is naturally given by $(X, K) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$ (i.e., the lift of $(x, k) \in C_0^{1-var}(\mathcal{V} \oplus \mathcal{W})$). This also extends to a continuous map

$$G\Omega_p(\mathcal{V}) \times C_0^{q-var}(\mathcal{W}) \ni (X, k) \mapsto (X, K) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W}).$$

In the following we briefly prove these two facts. Since they are very similar, we prove the latter case only. The first level path is clearly $(X_{s,t}^1, K_{s,t}^1)$. When, X, K are smooth rough paths, the second level path are given by

$$\left(X_{s,t}^2, \quad \int_s^t (x_u - x_s) \otimes dk_u, \quad \int_s^t (k_u - k_s) \otimes dx_u, \quad K_{s,t}^2 \right)$$

By using the Young integral, we see that these integrals naturally extend and that the second, the third, and the fourth components are of 1-variation finite.

Now we consider the third level path. $(1, 1, 1)$ and $(2, 2, 2)$-components are clearly $X_{s,t}^3$ and $K_{s,t}^3$, respectively. $(1, 1, 2)$-component is given by $\int_s^t X_{s,u}^2 \otimes dk_u$, which extends to the Young sense since $u \mapsto X_{s,u}^2$ is of finite p-variation by Chen's identity. $(1, 2, 1)$-component is given by

$$\int_s^t \int_s^{u_3} \int_s^{u_2} dx_{u_1} \otimes dk_{u_2} \otimes dx_{u_3} = \int_s^t \left(\int_s^{u_3} (x_{u_2} - x_s) \otimes dk_{u_2} \right) \otimes dx_{u_3}.$$

This also extends to Young integral (twice).

$(2, 1, 1)$-component is most complicated. If x, k is (piecewise) C^1, then

$$\int_s^t \int_s^{u_2} dk_{u_1} \otimes dX^2_{s, u_2} = \int_s^t \left(\int_s^{u_2} \int_s^{u_2} k'_{u_1} \otimes x'_\tau \otimes x'_{u_2} du_1 d\tau \right) du_2.$$

On the right hand side, integration is over $\{s < u_1 < u_2 < t, s < \tau < u_2 < t\}$. But, modulo Lebesgue measure-zero-set, this set is equal to

$$\{s < \tau < u_1 < u_2 < t\} \cup \{s < u_1 < \tau < u_2 < t\} \qquad \text{(disjoint union)}.$$

Therefore,

$$\int_s^t \int_s^{u_2} \int_s^\tau dk_{u_1} \otimes dx_\tau \otimes dx_{u_2}$$
$$= \int_s^t \int_s^{u_2} (k_{u_2} - k_s) \otimes dX^2_{s, u_2} - \int_s^t \left(\int_s^{u_2} dk_{u_1} \otimes (x_{u_1} - x_s) \right) \otimes dx_{u_2}.$$

In a similar way as above, integrals on the right hand side extend to the Young sense. For general x and k, one should just take limits.

The proof for remaining components, namely, $(1, 2, 2), (2, 1, 2), (2, 2, 1)$, are easier and omitted. Note that all the (i, j, l)-components except $(1, 1, 1)$ is of finite 1-variation.

8.2 The scale invariance of fractional Brownian rough paths

In this subsection we prove the scale invariance of fBRP. Since construction of fBRP is related to the dyadic partition of $[0, 1]$, this is not so obvious from the scale invariance of fBm.

Let $(w_t^H)_{0 \le t \le 1}$ be the d-dimensional fBm with Hurst parameter $H \in (1/4, 1/2]$ and let $0 < c \le 1$. It is well-known that $(c^{-H} w_{ct}^H)_{0 \le t \le 1}$ and w^H have the same law. A similar fact holds for the law of fBRP $W^H = (W_{s,t}^H)_{0 \le s \le t \le 1}$.

Proposition 8.1 *Let $H \in (1/4, 1/2]$. Then, for any $0 < c \le 1$, $(c^{-H} W_{cs, ct}^H)_{0 \le s \le t \le 1}$ and W^H have the same law.*

The rest of this subsection is devoted to proving of Proposition 8.1. Let us generalize the (semi)norm $D_{j,p}$. Fix an integer $K \ge 2$. For each sequence of integers $(\eta_1, \eta_2, \eta_3, \ldots)$ such that $2 \le \eta_i \le K$ for all i, we set $M_n = \eta_1 \eta_2 \cdots \eta_n$. We consider the following partitions of $[0, 1]$;

$$\mathcal{Q}_n = \{0 = t_0^n < t_1^n < \cdots < t_{M_n-1}^n < t_{M_n}^n = 1\}, \quad \text{where } t_k^n = k/M_n.$$

Clearly, $\mathcal{Q}_n \subset \mathcal{Q}_{n+1}$ for all n and the mesh tends to zero as $n \to \infty$. When $K = 2$ and $\eta_i = 2$ for all i, these are the dyadic partitions.

Given such a sequence $\vec{\mathcal{Q}} = \{\mathcal{Q}_n\}_{n=1,2,\dots}$ of partitions, we define (semi)norms by

$$D_{j,p}^{\vec{\mathcal{Q}}}(X,Y) = \Big(\sum_{n=1}^{\infty} n^{\gamma} \sum_{l=1}^{M_n} |X_{t_{l-1}^n, t_l^n}^j - Y_{t_{l-1}^n, t_l^n}^j|^{p/j} \Big)^{j/p} \tag{8.1}$$

for $\gamma > p-1$ and $1 \le j \le [p]$. When $Y = 0$, we write $D_{j,p}^{\vec{\mathcal{Q}}}(X) = D_{j,p}^{\vec{\mathcal{Q}}}(X,0)$.

By using these generalized (semi)norms, we can estimate p-variation norms as in (3.8). Note that the constant c in the next lemma does not depend on the particular choice of $\vec{\mathcal{Q}} = \{\mathcal{Q}_n\}$, but only on K.

Lemma 8.2 *Let $\vec{\mathcal{Q}} = \{\mathcal{Q}_n\}$ and $D_{j,p}^{\vec{\mathcal{Q}}}$ be as above. Then, there is a positive constant $c = c(K,p,\gamma)$ such that inequaliites in (3.8) hold for $D_{j,p}^{\vec{\mathcal{Q}}}$ and $c = c(K,p,\gamma)$, instead of $D_{j,p}$ and c_1.*

Proof. We omit a proof because we can prove this lemma by just modifying the dyadic case (pp. 61–67 [28]). ∎

For a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ and $w \in C_0(\mathbf{R}^d)$, we denote by $w(\mathcal{P})$ the piecewise linear approximation associated with respect to $\mathcal{P}$, i.e., $w(\mathcal{P})_{t_i} = w_{t_i}$ for all $t_i \in \mathcal{P}$ and linear on each subinterval $[t_{i-1}, t_i]$.

Lemma 8.3 *Let $1/4 < H \le 1/2$, $p < 1/H$ and $w^H = (w_t^H)_{0 \le t \le 1}$ be a d-dimensional fBm. Let $K \ge 2$ be an integer. Set $\eta_1 = K$ and $\eta_i = 2$ for all $i \ge 2$ and set $M_m = K 2^{m-1}$ and the partition $\mathcal{Q}_m$ as above. Then, $W^H(\mathcal{Q}_m)$ converges almost surely in $G\Omega_p(\mathbf{R}^d)$ and the limit $\lim_{m \to \infty} W^H(\mathcal{Q}_m)$ is the same as fBrp (i.e., the limit when $K = 2$) for any K.*

Proof. When $K = 2$, the partition is called dyadic and the definition of fBrp associated to this particular partition. Let $K \ge 3$. If we restrict $w^H(\mathcal{Q}_m)$ on each subinterval $[(i-1)/K, i/K]$ $(1 \le i \le K)$, then the same argument holds as in the case $K = 2$. One can verify convergence on the whole interval $[0,1]$ by using Chen's identity. Hence, we have only to show that the two almost sure limits are the same.

Set, for convenience,

$$\mathcal{Q}_m = \{0 < \frac{1}{2^m} < \frac{2}{2^m} < \cdots < \frac{2^m - 1}{2^m} < 1\},$$

$$\hat{\mathcal{Q}}_m = \{0 < \frac{1}{K \cdot 2^{m-1}} < \frac{2}{K \cdot 2^{m-1}} < \cdots < \frac{K \cdot 2^{m-1} - 1}{K \cdot 2^{m-1}} < 1\}.$$

We will show that $W^H(\mathcal{Q}_m)$ and $W^H(\hat{\mathcal{Q}}_{m+1})$ have the same limit. Now, for given m, set $\mathcal{R}_i = \mathcal{Q}_i$ for $1 \le i \le m$ and $\mathcal{R}_i = \hat{\mathcal{Q}}_i$ for $i > m$. In other words, $\vec{\mathcal{R}}(m) = (\mathcal{R}(m)_i)_{i=1,2,\dots}$ corresponds to $(\eta_i)_{i=1,2,\dots}$ such that $\eta_{m+1} = K$ and $\eta_i = 2$ for all $i \ne m+1$.

We will use Lemma 8.2 for this partition $\vec{\mathcal{R}}(m)$ and mimick the proofs in Sections 4.2–4.5, [28]. It is clear that the limits of the first level paths coincide. Moreover, it is easy to see that

$$\mathbb{E}\big[D_{1,p}^{\vec{\mathcal{R}}(m)}(W^H(\hat{\mathcal{Q}}_{m+1})^1, W^H(\mathcal{Q}_m)^1)^p\big] \leq C\Big(\frac{1}{2^m}\Big)^{(hp-1)/2} \tag{8.2}$$

$$\mathbb{E}\big[D_{1,p}^{\vec{\mathcal{R}}(m)}(W^H(\mathcal{Q}_m)^1)^p\big] + \mathbb{E}\big[D_{1,p}^{\vec{\mathcal{R}}(m)}(W^H(\hat{\mathcal{Q}}_{m+1})^1)^p\big] \leq C\sum_{n=1}^{\infty} n^\gamma \Big(\frac{1}{2^n}\Big)^{hp-1} \leq C \tag{8.3}$$

for some positive constants $C = C(K, p, \gamma, d)$, which may depend only on K, p, γ, d. Here, and below, the positive constant C may change from line to line.

Now we consider the second level paths. Let $n \geq m$. Then, for $j = 2, 3$,

$$W^H(\mathcal{R}_m)_{t_{k-1}^n, t_k^n}^j = \frac{1}{j!}\Big(\frac{M(m)_m}{M(m)_n}\Big)^j [W_{t_{l-1}^m, t_l^m}^{H,1}]^{\otimes j}, \tag{8.4}$$

where $(t_k^n)_k$ and $(t_l^m)_l$ are partition points of $\mathcal{R}(n)$ and $\mathcal{R}(m)$, respectively, and k and l are such that $[t_{k-1}^n, t_k^n] \subset [t_{l-1}^m, t_l^m]$. By using this, we can show the following in the same way as in p.77, [28];

$$\sum_{k=1}^{M(m)_n} \mathbb{E}\Big[\big|W^H(\hat{\mathcal{Q}}_{m+1})_{t_{k-1}^n, t_k^n}^2 - W^H(\mathcal{Q}_m)_{t_{k-1}^n, t_k^n}^2\big|^{p/2}\Big] \leq C\Big(\frac{1}{2^{m+n}}\Big)^{(ph-1)/2}. \tag{8.5}$$

The case $n < m$ is a little bit more complex. Set $I_k^n = [t_{k-1}^n, t_k^n]$ and $\triangle_l^{m+1} w^H = w_{t_l^{m+1}}^H - w_{t_{l-1}^{m+1}}^H = W^H(\hat{\mathcal{Q}}_{m+1})_{t_{l-1}^{m+1}, t_l^{m+1}}^1$, etc. As in Lemma 4.2.1, p. 71, [28], we can show that

$$
\begin{aligned}
&W^H(\hat{\mathcal{Q}}_{m+1})_{t_{k-1}^n, t_k^n}^2 - W^H(\mathcal{Q}_m)_{t_{k-1}^n, t_k^n}^2 \\
&= \frac{1}{2}\sum_{s; I_s^m \subset I_k^n}\Big[\sum_{r<l; I_r^{m+1}, I_l^{m+1} \subset I_s^m}\Big(\triangle_r^{m+1} w^H \otimes \triangle_l^{m+1} w^H - \triangle_l^{m+1} w^H \otimes \triangle_r^{m+1} w^H\Big)\Big]
\end{aligned}
$$

From this and pp.79–82, [28], we see that

$$\mathbb{E}\Big[\big|W^H(\hat{\mathcal{Q}}_{m+1})_{t_{k-1}^n, t_k^n}^2 - W^H(\mathcal{Q}_m)_{t_{k-1}^n, t_k^n}^2\big|^{p/2}\Big] \leq C 2^{(m-n)p/4}\Big(\frac{1}{2^m}\Big)^{hp},$$

from which we see that (8.5) holds for $m > n$, too. From (8.5), we obtain

$$\mathbb{E}\big[D_{2,p}^{\vec{\mathcal{R}}(m)}(W^H(\hat{\mathcal{Q}}_{m+1})^2, W^H(\mathcal{Q}_m)^2)^{p/2}\big] \leq C\Big(\frac{1}{2^m}\Big)^{(hp-1)/2} \tag{8.6}$$

Combining this with Lemma 8.2 and (8.2), (8.3), we obtain that

$$\mathbb{E}[\|W^H(\hat{\mathcal{Q}}_{m+1})^2 - W^H(\mathcal{Q}_m)^2\|_{p/2-var}^{p/2}] \to 0 \qquad \text{as } m \to \infty.$$

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Note that, in the same way as in (8.6), we can prove that, for any m' ($1 \leq m' \leq m-1$), the following inequality holds;

$$\mathbb{E}\big[D_{2,p}^{\vec{\mathcal{R}}(m)}(W^H(\mathcal{R}_{m'+1})^2, W^H(\mathcal{R}_{m'})^2)^{p/2}\big]^{2/p} \leq C\Big(\frac{1}{2^{m'}}\Big)^{(hp-1)/p}$$

From this and the triangle inequality,

$$\mathbb{E}\big[D_{2,p}^{\vec{\mathcal{R}}(m)}(W^H(\mathcal{Q}_m)^2)^{p/2}\big] + \mathbb{E}\big[D_{2,p}^{\vec{\mathcal{R}}(m)}(W^H(\hat{\mathcal{Q}}_{m+1})^2)^{p/2}\big] \leq C \tag{8.7}$$

Finally, we consider the third level paths. In this case, $3 \leq 1/H < p < 4$. When $n \geq m$, we can use (8.4) for $j = 3$ to obtain

$$\sum_{k=1}^{M(m)_n} \mathbb{E}\Big[\big|W^H(\hat{\mathcal{Q}}_{m+1})^3_{t^n_{k-1}, t^n_k} - W^H(\mathcal{Q}_m)^3_{t^n_{k-1}, t^n_k}\big|^{p/3}\Big] \leq C\Big(\frac{1}{2^{m+n}}\Big)^{(ph-1)/2} \tag{8.8}$$

for $n \geq m$.

For $n < m$, first note that, as in p.72, [28],

$$\begin{aligned}
W^H(\mathcal{Q}_m)_{t^n_{k-1}, t^n_k} = {}& \frac{1}{6}(\triangle^n_k w^H)^{\otimes 3} + \frac{1}{3}\sum_{r<l} \triangle^m_r w^H \otimes \triangle^m_l w^H \otimes \triangle^m_l w^H \\
& + \frac{1}{3}\sum_{r<l} \triangle^m_r w^H \otimes \triangle^m_r w^H \otimes \triangle^m_l w^H \\
& - \frac{1}{6}\sum_{r>l} \triangle^m_r w^H \otimes \triangle^m_r w^H \otimes \triangle^m_l w^H \\
& - \frac{1}{6}\sum_{r>l} \triangle^m_r w^H \otimes \triangle^m_l w^H \otimes \triangle^m_l w^H \\
& - \frac{1}{6}\sum_{r\neq l} \triangle^m_l w^H \otimes \triangle^m_r w^H \otimes \triangle^m_l w^H \\
& + \sum_{r<u<l} \triangle^m_r w^H \otimes \triangle^m_u w^H \otimes \triangle^m_l w^H \\
& - \frac{1}{6}\sum_{r,u,l;\text{distinct}} \triangle^m_r w^H \otimes \triangle^m_u w^H \otimes \triangle^m_l w^H,
\end{aligned}$$

where the indices run over all r, u, l such that $I^m_r, I^m_u, I^m_l \subset I^n_k$ additionally holds. A similar equality holds for $W^H(\hat{\mathcal{Q}}_{m+1})_{t^n_{k-1}, t^n_k}$, too.

Now we set several motations. Set $K_1 = K_1(k)$ by

$$K_1 = \frac{1}{3}\sum_{l;I^m_l \subset I^n_k}\Big(\sum_{\xi<\xi';I^{m+1}_\xi, I^{m+1}_{\xi'}\subset I^m_l} \triangle^{m+1}_\xi w^H \otimes \triangle^{m+1}_{\xi'} w^H \otimes \triangle^{m+1}_{\xi'} w^H\Big),$$

where the sum inside runs over all ξ, ξ' such that $\xi < \xi'$ and $I_\xi^{m+1} \subset I_l^m, I_{\xi'}^{m+1} \subset I_l^m$ for a given l. There are $K(K-1)/2$ such pairs of $(\xi < \xi')$. Simlarly,

$$K_2 = \frac{1}{3} \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \triangle_\xi^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H \right),$$

$$K_3 = -\frac{1}{6} \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \triangle_{\xi'}^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \right),$$

$$K_4 = -\frac{1}{6} \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \right),$$

$$K_5 = -\frac{1}{6} \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \triangle_{\xi'}^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \right),$$

$$K_6 = -\frac{1}{6} \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \triangle_{\xi'}^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H \right).$$

In a simlar way, we set

$$K_7 = \frac{1}{2} \sum_{r < l; I_r^m, I_l^m \subset I_k^n} \Bigg[\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \\ \triangle_r^m w^H \otimes \left(\triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H - \triangle_{\xi'}^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \right) \Bigg],$$

$$K_8 = \frac{1}{2} \sum_{r < l; I_r^m, I_l^m \subset I_k^n} \Bigg[\sum_{\xi < \xi'; I_\xi^{m+1}, I_{\xi'}^{m+1} \subset I_l^m} \\ \left(\triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H - \triangle_{\xi'}^{m+1} w^H \otimes \triangle_\xi^{m+1} w^H \right) \otimes \triangle_r^m w^H \Bigg],$$

$$K_9 = \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi < \xi' < \xi''; I_\xi^{m+1}, I_{\xi'}^{m+1}, I_{\xi''}^{m+1} \subset I_l^m} \triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H \otimes \triangle_{\xi''}^{m+1} w^H \right),$$

$$K_{10} = -\frac{1}{6} \sum_{l; I_l^m \subset I_k^n} \left(\sum_{\xi, \xi', \xi''; \text{distinct}} \triangle_\xi^{m+1} w^H \otimes \triangle_{\xi'}^{m+1} w^H \otimes \triangle_{\xi''}^{m+1} w^H \right),$$

where in the sum on the right hand side of K_{10}, the indices run over all distinct ξ, ξ', ξ'' such that $I_\xi^{m+1}, I_{\xi'}^{m+1}, I_{\xi''}^{m+1} \subset I_l^m$.

Then, in the same way as in pp.72–73, [28], (which, however, is not so easy as it may seem), we obtain the following;

$$W^H(\hat{\mathcal{Q}}_{m+1})_{t_{k-1}^n, t_k^n}^3 - W^H(\mathcal{Q}_m)_{t_{k-1}^n, t_k^n}^3 = K_1 + \cdots + K_{10}.$$

Note that K_9, K_{10} do not appear in the dyadic case (i.e., the case $K = 2$).

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In the same way as in pp. 86–88, [28] (although it contains a few typos, it is basically correct), we have the following estimates;

$$\mathbb{E}\big[|K_i|^2\big] \;\leq\; C2^{m-n}\Big(\frac{1}{2^m}\Big)^{6h}, \qquad (1 \leq i \leq 6) \tag{8.9}$$

$$\mathbb{E}\big[|K_i|^2\big] \;\leq\; C2^{(m-n)(1+2h)}\Big(\frac{1}{2^m}\Big)^{6h}, \qquad (7 \leq i \leq 8) \tag{8.10}$$

By the Wick formula for six centered Gaussian random variables and eq. (4.53) in p. 85, [28], we can estimate the moments of K_9, K_{10} just like that of K_1 and obtain (8.9) for $i = 9, 10$, too. From (8.9), (8.10) and the fact that $ph - p + 3 > 0$, we obtain (8.8) for the case $n < m$, too.

From (8.8), Lemma 6.2, (8.3), (8.7), we see that

$$\mathbb{E}\big[D_{3,p}^{\vec{\mathcal{R}}(m)}(W^H(\hat{\mathcal{Q}}_{m+1})^3, W^H(\mathcal{Q}_m)^3)^{p/3}\big]^{3/p} \;\leq\; C\Big(\frac{1}{2^m}\Big)^{3(hp-1)/2p}$$

and that
$$\mathbb{E}[\|W^H(\hat{\mathcal{Q}}_{m+1})^3 - W^H(\mathcal{Q}_m)^3\|_{p/3-var}^{p/3}] \to 0 \qquad \text{as } m \to \infty.$$

This completes the proof. ∎

Proof of Proposition 8.1   It is sufficient to prove the proposition for rational c. We may assume $c = L/K$ for positive integers K, L with $K > L$.

First, instead of $[0, 1]$, let us regard $[0, c]$ as the whole interval. Let

$$\mathcal{P}_n \;=\; \{0 < \frac{c}{2^n} < \frac{2c}{2^n} < \cdots < \frac{(2^n - 1)c}{2^n} < c\},$$

$$\hat{\mathcal{P}}_n \;=\; \{0 < \frac{c}{L \cdot 2^{n-1}} < \frac{2c}{L \cdot 2^{n-1}} < \cdots < \frac{(L \cdot 2^{n-1} - 1)c}{L \cdot 2^{n-1}} < c\}.$$

Then, by using a trivial modification of Lemma 8.3, we easily see that

$$\lim_{n\to\infty} W^H(\mathcal{P}_n) = \lim_{n\to\infty} W^H(\hat{\mathcal{P}}_n), \qquad \text{on } [0, c]\text{-interval.}$$

Second, we use Lemma 8.3 on the interval $[0, 1]$ for the partitions

$$\mathcal{Q}_n \;=\; \{0 < \frac{1}{2^n} < \frac{2}{2^n} < \cdots < \frac{2^n - 1}{2^n} < 1\},$$

$$\hat{\mathcal{Q}}_n \;=\; \{0 < \frac{1}{K \cdot 2^{n-1}} < \frac{2}{K \cdot 2^{n-1}} < \cdots < \frac{K \cdot 2^{n-1} - 1}{K \cdot 2^{n-1}} < 1\}$$

to obtain

$$(W^H :=) \lim_{n\to\infty} W^H(\mathcal{Q}_n) = \lim_{n\to\infty} W^H(\hat{\mathcal{Q}}_n), \qquad \text{on } [0, 1]\text{-interval.}$$

Now, observe that $\hat{\mathcal{P}}_n$ and $\hat{\mathcal{Q}}_n$ are the same partition. Combining them all, we have

$$W^H = \lim_{n \to \infty} W^H(\mathcal{P}_n), \qquad \text{on } [0, c]\text{-interval.}$$

Set $\tilde{w}_t^H = c^{-H} w_{ct}^H$. It is well-known that $\tilde{w}^H$ is again a d-dimensional fBm. We consider the dyadic approximation of $\tilde{w}^H$.

$$\lim_{n \to \infty} \tilde{W}^H(\mathcal{Q}_n) = \lim_{n \to \infty} \left(c^{-H} W^H(\mathcal{P}_n)_{cs,ct}\right)_{0 \leq s \leq t \leq 1} = \left(c^{-H} W_{cs,ct}^H\right)_{0 \leq s \leq t \leq 1}.$$

As is explained, the left hand side is fBrp. Note that we used the following fact. For $x \in C_0^{1-var}(\mathbf{R}^d)$, set $x(c)_t = x_{ct}$. It is easy to see that $X_{cs,ct} = X(c)_{s,t}$, where the right hand side is the lift of the time-changed path $x(c)$. Thus, we have shown Proposition 8.1.

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